Some solutions to Problem Set 4.

---

1. Since $\mathbb{R}$ equipped with the usual metric is complete, it suffices to show that the sequence $\{d(x_n, y_n)\}$ is Cauchy. Let $\varepsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have

$$d(x_n, x_m) < \varepsilon/2 \quad \text{and} \quad d(y_n, y_m) < \varepsilon/2$$

By the triangle inequality

$$|d(x_n, y_n) - d(x_m, y_m)| \leqslant d(x_n, x_m) + d(y_n, y_m).$$

Hence for $n, m \geq N$,

$$|d(x_n, y_n) - d(x_m, y_m)| \leqslant d(x_n, x_m) + d(y_n, y_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

So $(d(x_n, y_n)\}$ is Cauchy and since $\mathbb{R}$ is complete, $(d(x_n, y_n)\}$ converges.

2. Let $m > n \geq k$. Then

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leqslant \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \leqslant \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}} \leqslant \frac{1}{2^{k-1}}. \end{aligned}$$

So if $\varepsilon > 0$ choose $k \in \mathbb{N}$ such that $1/2^{k-1} < \varepsilon$. Then, in view of the above inequality,

$$d(x_n, x_m) < \frac{1}{2^{k-1}} < \varepsilon$$

for all $n, m \geq k$. Hence (x_n) is Cauchy.

3. (a) The space $((0, \infty), d)$ is not complete. For example, the sequence $\{x_n\}$ with $x_n = \frac{1}{\sqrt{n}}$ is Cauchy but doesn't converge in this space. Indeed, if $x \in (0, \infty)$ and $d(x_n, x) \to 0$, then taking $\varepsilon = x^2/2$, then there is N such that $|x^2 - 1/n| < x^2/2$ for $n \geq N$. Then taking a limit as $n \to \infty$ we get $x^2 \leqslant x^2/2$ showing that $x = 0$, contradiction.
(b) Let $(x_n) \in (-\pi/2, \pi/2)$ be a Cauchy sequence with respect to d. Then, in view of the definition of d, the sequence $y_n = \tan x_n$ is Cauchy in $\mathbb{R}$ with the standard metric. Hence there is $y \in \mathbb{R}$ such that $|y_n - y| \to 0$. Let $x = \tan^{-1} y \in (-\pi/2, \pi/2)$. Then

$$d(x_n, x) = |\tan x_n - \tan x| = |y_n - y| \to 0$$

and so $((-\pi/2, \pi/2), d)$ is complete.

4. The sequence (x_n) with $x_n = 1/n$ is Cauchy in $((0, 1], d)$ but does not converge in this space, so $((0, 1], d)$ is not complete. Let (x_n) be a Cauchy sequence in $((0, 1], \overline{d})$. This implies that the sequence (y_n) with $y_n = 1/x_n$ is Cauchy in $[1, \infty)$ equipped with the standard metric and since this space is complete there is $y \in [1, \infty)$ such that $|y_n - y| \to 0$. Setting $x = 1/y$ we get

$$\overline{d}(x_n, x) = \left| \frac{1}{x_n} - \frac{1}{x} \right| = |y_n - y| \to 0$$

showing that $((0, 1], \overline{d})$ is complete. Equivalence of d and $\overline{d}$ follows from the continuity of the function $f : (0, 1] \to [1, \infty)$, $f(x) = 1/x$ and its inverse $f^{-1} : [1, \infty) \to (0, 1]$ considered with the usual metrics.

5. Suppose that (X, d) is complete and (y_n) is Cauchy in $(Y, \widetilde{d})$. We have to show that there exists $y \in Y$ such that $\widetilde{d}(y_n, y) \to 0$ as $n \to \infty$. For every n there exists $x_n \in X$ such that $f(x_n) = y_n$. We claim that (x_n) is Cauchy. Indeed, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$d(f^{-1}(a), f^{-1}(b)) < \varepsilon \text{ for all } a, b \in Y \text{ satisfying } \widetilde{d}(a, b) < \delta \tag{1}$$

since f^{-1} is uniformly continuous. Since (y_n) is Cauchy in $(Y, \widetilde{d})$, there is N such that

$$\widetilde{d}(y_n, y_m) < \delta \text{ for all } n, m > N. \tag{2}$$

Combining (1) and (2), we get

$$d(x_n, x_m) = d(f^{-1}(y_n), f^{-1}(y_m)) < \varepsilon$$

for $n, m > N$. So (x_n) is Cauchy and since (X, d) is complete, there is $x \in X$ such that $d(x_n, x) \to 0$, and since f is continuous, $\widetilde{d}(y_n, f(x)) = \widetilde{d}(f(x_n), f(x)) \to 0$. So $(Y, \widetilde{d})$ is complete.

6. (a) For every n choose a point in F_n and call it x_n. Claim: (x_n) is Cauchy. Indeed, take $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\text{diam}F_k < \varepsilon$, This is possible since $\text{diam}F_n \to 0$. Since $x_n, x_m \in F_k$ for $n, m \geq k$,

$$d(x_n, x_m) \leqslant \sup\{d(x, y) |\ x, y \in F_k\} = \text{diam}F_k < \varepsilon.$$

Thus (x_n) is Cauchy and since (X, d) is complete, there exists $x \in X$ such that $d(x_n, x) \to 0$. We claim that $x \in \bigcap_{n\in\mathbb{N}} F_n$. Take any $k \in \mathbb{N}$, then $x_n \in F_k$ for all $n \geq k$. Since $x_n \to x$ and F_k is closed, $x \in F_k$. Hence $x \in \bigcap_{n\in\mathbb{N}} F_n$. If $y \in \bigcap_{n\in\mathbb{N}} F_n$ and $x \neq y$, then

$$0 < d(x, y) \leqslant \text{diam}F_k \to 0,$$

contradiction.

(b) (i) Consider $X = (0, 1]$ with the usual metric d. Then (X, d) is not complete. Take $F_n = (0, 1/n]$. Then F_n is closed in X, $F_{n+1} \subset F_n$ for all $n \geq 1$, and $\text{diam}F_n = 1/n \to 0$. However, $\bigcap_{n\geq 1} F_n = \emptyset$.

(ii) In $\mathbb{R}$ with the usual metric d consider $F_n = (0, 1/n]$. Then $\text{diam}F_n = 1/n \to 0$ and F_n is not closed in $\mathbb{R}$. The intersection $\bigcap_{n\geq 1} F_n = \emptyset$.

(iii) In $\mathbb{R}$ with the usual metric d consider $F_n = [n, \infty)$. Then F_n is closed in $\mathbb{R}$, $F_{n+1} \subset F_n$ but $\text{diam}F_n = \infty$, and $\bigcap_{n\geq 1} F_n = \emptyset$.

(c) Conversely, let (x_n) be Cauchy. Set $A_n = \{x_k |\ k \geq n\}$ and $F_n = \overline{A_n}$. Then each F_n is closed and $F_{n+1} \subset F_n$. If $\varepsilon > 0$, then there is N such that $d(x_k, x_m) < \varepsilon$ for $k, m \geq N$. So diam $A_n \leqslant \varepsilon$ for all $n \geq N$. Since diam F_n = diam $\overline{A_n}$ = diam A_n, it follows that diam $F_n \to 0$. Thus, by assumption, there is $x \in \bigcap F_n$. Claim: (x_n) converges to x. From $x \in \bigcap F_n$ and the definition of the closure it follows that there exists an increasing sequence of integers $k_1 < k_2 < \cdots$ such that $x_{k_n} \in A_{k_n}$ and $d(x, x_{k_n}) < 1/n$. So the subsequence (x_{k_n}) converges and since (x_n) is Cauchy, the sequence (x_n) converges to x. So X is complete.

7. Arguing by contradiction we assume that there is no point x^* having the properties as above. Take any point of X and call it x_1. Then there is a point x_2 such that $f(x_2) > 2f(x_1)$ and $x_2 \in B(x_1, \frac{1}{\sqrt{f(x_1)}})$. For the point x_2 we find x_3 such that $f(x_3) > 2f(x_2)$ and $x_3 \in B(x_2, \frac{1}{\sqrt{f(x_2)}})$. Continuing in this way we obtain a sequence (x_n) satisfying $f(x_{n+1}) > 2f(x_n)$ and $d(x_{n+1}, x_n) < \frac{1}{\sqrt{x_n}}$ for all $n \geq 1$. Observe that if $a = f(x_1) > 0$, then $f(x_n) > 2^{n-1}a$. Hence for $n < m$

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, d_{n+1}) + d(x_{n+1}, x_{n+1}) + \cdots d(x_{m-1}, x_m) \\ &\leqslant \frac{1}{\sqrt{f(x_n)}} + \frac{1}{\sqrt{f(x_{n+1})}} + \cdots \leqslant \frac{\sqrt{2}}{\sqrt{a}} \left[\frac{1}{(\sqrt{2})^n} + \frac{1}{(\sqrt{2})^{n+1}} + \cdots \right] \\ &\leqslant \frac{2}{\sqrt{a}(\sqrt{2} - 1)} \cdot \frac{1}{(\sqrt{2})^n} \end{aligned}$$

and since $1/(\sqrt{2})^n \to 0$, it follows that (x_n) is Cauchy. Since X is complete there is x such that $d(x_n, x) \to 0$. Now we obtain a contradiction by noticing that $f(x_n) > 2^{n-1}a \to \infty$ as $n \to \infty$, but we also have $f(x_n) \to f(x) \in \mathbb{R}$ since f is continuous.