Some solutions to Problem Set 5.

1. (a) Not a contraction. For example, \( d(f(0), f(-1)) = e - 1 > d(0, -1) = 1 \).
(b) Not a contraction. Indeed if \( f \) is a contraction, then \( \|f(x) - f(y)\| \leq c|x - y| \) for some \( c < 1 \) and all \( x, y \geq 0 \). On the other hand, the mean value theorem implies that for every \( y > 0 \) there is \( z := z_y \in (0, y) \) such that \( |f(0) - f(y)| = e^{-z}|y| \leq c|y| \) so that \( e^{-z} < c \). But if \( y \to 0 \), then \( z \to 0 \) and \( e^{-z} \to 1 \), contradiction.
(c) \( f \) is a contraction by the mean value theorem, since \( |f'(c)| = |-e^{-y}| \leq e^{-1} \) for all \( c \geq 0 \).
(d) \( f \) is not a contraction since if \( |f(x) - f(y)| \leq c|x - y| \) with \( c < 1 \), then, in particular, \( |f(x/2) - f(y)| = |f(y)| \leq c|x/2 - y| \). By the mean value theorem, for every \( y > \pi/2 \), there is \( z \in (\pi/2, y) \) so that \( |f(y)| = |f(\pi/2) - f(y)| = |\sin(z)| \cdot |y - \pi/2| \). Hence \( |\sin(z)| \leq c \). But if \( y \to \pi/2 \), then \( z \to \pi/2 \) and so \( \sin z \to 1 \), contradiction.
(e) \( f \) is a contraction. First, \( \cos : \mathbb{R} \to \mathbb{R} \) satisfies \( |\cos x - \cos y| \leq |x - y| \) and \( \cos(\mathbb{R}) = [-1, 1] \). Considering \( x \mapsto \cos x \) on the set \([-1, 1]\), it is a contraction with the constant \( \sin 1 < 1 \). So \( |\cos(\cos x) - \cos(\cos y)| \leq (\sin 1) \cdot |\cos x - \cos y| \leq (\sin 1) \cdot |x - y| \).

2. \( f \) is a contraction with respect to \( d_1 \) but not with respect to \( d_2 \) and \( d_\infty \). Take \( x = (1, 1) \) and \( y = (0, 0) \). Then \( f(1, 1) = (8/5, 1/5) \) and \( f(0, 0) = (0, 0) \). Then, \( d_2((1, 1), (0, 0)) = \sqrt{2} \) and

\[
d_2((f(1, 1), f(0, 0)) = d_2((8/5, 1/5), (0, 0)) = \sqrt{65}/25 = \sqrt{13}/10 \cdot \sqrt{2} = \sqrt{13}/10 \cdot d_2((1, 1), (0, 0)).
\]

Similarly, \( d_\infty((1, 1), (0, 0)) = 1 \) and

\[
d_\infty((f(1, 1), f(0, 0)) = d_\infty((8/5, 1/5), (0, 0) = \max\{8/5, 1/5\} = 8/5 = 8/5 \cdot d_\infty((1, 1), (0, 0)).
\]

For the \( d_1 \) metric we have

\[
d_1(f(x_1, y_1), f(x_2, y_2)) = d_1\left(\frac{1}{10}(8x_1 + 8y_1, x_1 + y_1), \frac{1}{10}(8x_2 + 8y_2, x_2 + y_2)\right) = \frac{8}{10} |(x_1 + y_1) - (x_2 + y_2)| + \frac{1}{10} |(x_1 + y_1) - (x_2 + y_2)| = \frac{9}{10} |(x_1 - x_2) + (y_1 - y_2)| = \frac{9}{10} d_\infty((x_1, y_1), (x_2, y_2))
\]

so \( f \) is a contraction with respect to the metric \( d_1 \) with contraction constant \( 9/10 \).

3. (a) For \( x, y \in (0, a] \),

\[
|x^2 - y^2| = |x + y| \cdot |x - y| \leq \sqrt{|x| + |y|} \cdot |x - y| = (2a) \cdot |x - y|
\]

so that \( f \) is a contraction if \( a < 1/2 \). Observe that if \( x \in (0, a] \) with \( a < 1/2 \), then clearly \( 0 \leq x^2 \leq a^2 < a \) so that \( f : (0, a] \to (0, a] \). If \( x^2 = x \), then \( x = 0 \) or \( x = 1 \). Hence \( f : X \to X \) does not have a fixed point in \( X \). Note that \((X, d)\) is not complete.
(b) Clearly, \( x + \frac{1}{x} \geq 1 \) for \( x \geq 1 \) so that \( f : X \to X \). Observe next that

\[
\left( x + \frac{1}{x} \right) - \left( y + \frac{1}{y} \right) = \left( 1 - \frac{1}{xy} \right) \cdot (x - y)
\]

so that

\[
|f(x) - f(y)| = \left( 1 - \frac{1}{xy} \right) \cdot |x - y| < |x - y|
\]

since \( 1 - 1/(xy) < 1 \) for all \( x, y \geq 1 \). Assume that \( f \) has a fixed point \( x \). Then \( x + \frac{1}{x} = x \), so \( \frac{1}{x} = 0 \), contradiction. Note that \( f \) is not a contraction since otherwise \( 1 - 1/(xy) \leq \alpha \) for some constant \( \alpha < 1 \) and all \( x, y \geq 1 \). Taking \( y = 1 \) and \( x \) large we get a contradiction.
4. The space \((\mathcal{B}(x_0, r_0), d)\) is complete (as it is a closed subspace of a complete metric space) so it suffices to show that \(f : (\mathcal{B}(x_0, r_0)) \rightarrow \mathcal{B}(x_0, r_0)\). If \(x \in \mathcal{B}(x_0, r_0)\), then
\[
d(f(x), x_0) \leq d(f(x), f(x_0)) + d(f(x_0), x_0) \leq \alpha d(x, x_0) + r_0(1 - \alpha) \\
\leq \alpha r_0 + (1 - \alpha)r_0 = r_0.
\]
Hence \(f : \mathcal{B}(x_0, r_0) \rightarrow \mathcal{B}(x_0, r_0)\) is a contraction, and \(f\) has a unique fixed point in \(\mathcal{B}(x_0, r_0)\) by the Banach fixed point theorem.

5 (a) Observe that \([f(x)]^3 - e^x[f(x)]^2 + \frac{f(x)}{2} = e^x\) if and only if
\[
e^x + \frac{1}{2} \frac{f(x)}{1 + f(x)^2} = f(x)\text{ for all } x \in [0, 1].
\]
Thus, \(f\) is the fixed point of \(T : C[0, 1] \rightarrow C[0, 1]\) where
\[
(Tf)(x) = e^x + \frac{1}{2} \frac{f(x)}{1 + f(x)^2}.
\]
By the mean value theorem applied to the function \(h(t) = \frac{t}{1 + t^2}\), there exists \(c \in (a, b)\) such that
\[
\left| \frac{a}{1 + a^2} - \frac{b}{1 + b^2} \right| = \frac{1 - c^2}{(1 + c^2)^2} \cdot |a - b| \leq |a - b|.
\]
Thus, if \(f, g \in C[0, 1]\), then
\[
|(Tf)(x) - (Tg)(x)| = \left| \frac{1}{2} \frac{f(x)}{1 + f(x)^2} - \frac{g(x)}{1 + g(x)^2} \right| \leq \frac{1}{2} \cdot |f(x) - g(x)|
\]
for all \(x \in [0, 1]\). Consequently, if \(d_\infty\) is the supremum metric,
\[
d_\infty((Tf), (Tg)) \leq \frac{1}{2} \cdot d_\infty(f, g)
\]
so that \(T : C[0, 1] \rightarrow C[0, 1]\) is a contraction. Since \((C[0, 1], d_\infty)\) is complete, there is exactly one \(f\) such that \(Tf = f\).

(b) For \(t \in [0, a]\) we have
\[
|(Tf)(t) - (Tg)(t)| \leq \int_0^t |f(s) - g(s)| ds \leq a \cdot d_\infty(f, g).
\]
Hence \(d_\infty(Tf, Tg) \leq a \cdot d_\infty(f, g)\) and so \(T\) is a contraction. Hence there exists exactly one \(f\) such that \(Tf = f\). That is,
\[
f(t) = \sin t + \int_0^t f(s) ds, \quad t \in [0, a].
\]
Evaluating at \(t = 0\) we get \(f(0) = 0\). Differentiating both sides we get \(f'(t) = \cos t + f(t)\) for \(t \in [0, a]\). Multiplying by \(e^{-t}\) this is equivalent to \([e^{-t}f(t)]' = e^{-t} \cos t\). Integrating from \(0\) to \(s\) we get \(e^{-s}f(s) = \int_0^s e^{-t} \cos t dt = 1/2 + e^{-s}[\sin s - \cos s]/2\) which implies that \(f(t) = e^t/2 + [\sin t - \cos t]/2\).

(c) Define \(T : C[0, \pi] \rightarrow C[0, \pi]\) by
\[
(Tf)(t) = \frac{1}{3} \int_0^t \sin(t - s) \cdot f(s) ds.
\]
Now for \(0 \leq s \leq t \leq \pi\) we have \(\sin(t - s) \geq 0\), hence
\[
|(Tf)(t) - (Tg)(t)| \leq \frac{1}{3} \int_0^t |\sin(t - s)| |f(s) - g(s)| ds \leq \frac{1}{3} \int_0^\pi |\sin(t - s)| d(f, g) ds = \frac{2}{3} d(f, g),
\]
where \(d\) is the sup metric. Thus \(T\) is a contraction with contraction constant \(2/3\). So there is exactly one \(f\) in \(C[0, \pi]\) such that \(3f(t) = \int_0^t \sin(t - s) f(s) ds\) for all \(t \in [0, \pi]\). Since the zero function solves this equation, \(f \equiv 0\).
(d) The equation is equivalent to
\[ g(x) - \int_{0}^{1} e^{x-y-1} f(y) dy = f(x) \quad \text{for all } x \in [0, 1]. \]
So setting
\[ (Tf)(x) = g(x) - \int_{0}^{1} e^{x-y-1} f(y) dy \quad \text{for } x \in [0, 1], \]
we have to show that there is a fixed point of \( T \). We equip \( C[0,1] \) with the metric \( d(f,h) = \sup\{e^{-x}|f(x) - h(x)| : x \in [0,1]\} \). Then \( (C[0,1],d) \) is complete. Since
\[
|Tf(x) - Th(x)| \leq \int_{0}^{1} e^{x-y-1} f(y) - h(y) \, dy = e^{x} \cdot e^{-1} \int_{0}^{1} e^{-y} |f(y) - h(y)| \, dy \\
\leq e^{x} \cdot e^{-1} \cdot d(f,h)
\]
we conclude \( d(Tf, Th) \leq e^{-1} d(f,h) \). By Banach fixed point theorem there is exactly one \( f \) such that \( Tf = f \).

6. For \( k \geq 1 \) let \( U_{k} = \{ x \in \mathbb{R}^{n} | f_{k}(x) \neq 0 \} = f_{k}^{-1}(\mathbb{R} \setminus \{ 0 \}) \). Since \( \mathbb{R} \setminus \{ 0 \} \) is open and \( f_{k} \) is continuous (a linear map from a finite dimensional vector space to another vector space), we conclude that \( U_{k} \) is open. We claim that every \( U_{k} \) is dense. If not, then for some \( k \), there exists \( B(a,r) \subset \mathbb{R}^{n} \) such that \( B(a,r) \cap U_{k} = \emptyset \). Hence, for every \( x \in B(a,r) \), \( f_{k}(x) = 0 \). Take any \( x \in \mathbb{R}^{n} \) not equal to \( a \). Then \( y := \frac{r(x-a)}{2\|x-a\|} \in B(a,r) \) and so \( f(y) = 0 \). But since \( f_{k} \) is linear,
\[
0 = f(y) = \frac{r}{\|x-a\|} f(x) - \frac{r}{\|x-a\|} f(a) = \frac{r}{\|x-a\|} f(x)
\]
so that \( f_{k}(x) = 0 \). Hence \( f_{k} = 0 \) contradicting the assumption. Hence \( U_{k} \) is dense and open. In view of the Baire’s theorem, \( \bigcap_{k \geq 1} U_{k} \) is dense, in particular, non-empty. Take any \( x \in \bigcap_{k \geq 1} U_{k} \) and the result follows.

7. Arguing by contradiction assume that \( \{ f_{n}(x) \} \) is bounded for every irrational \( x \). For \( k \geq 1 \) define
\[
F_{k} = \{ x \in \mathbb{R} | \| f_{n}(x) \| \leq k \text{ for all } n \in \mathbb{N} \} = \bigcap_{n \in \mathbb{N}} f_{n}^{-1}([-k,k]).
\]
The set \( f_{n}^{-1}([-k,k]) \) is closed since \( f_{n} \) is continuous, and so \( F_{k} \) is closed. Moreover, \( \mathbb{Q}^{c} \subset \bigcup_{k \in \mathbb{N}} F_{k} \) since, by assumption, \( \{ f_{n}(x) \} \) is bounded for every irrational \( x \). So
\[
\mathbb{R} = \bigcup_{k \in \mathbb{N}} F_{k} \cup \bigcup_{r \in \mathbb{Q}} \{ r \},
\]
that is, \( \mathbb{R} \) is a countable union of closed sets since \( \mathbb{Q} \) is countable. Every singleton \( \{ r \} \) is nowhere dense. Consequently, in view of Baire theorem, some \( F_{k} \) has nonempty interior. Say \( F_{k}^{0} \neq \emptyset \). So \( (a,b) \subset F_{k} \), and \( \{ f_{n}(x) \} \) is bounded for any \( x \in (a,b) \). In particular, \( \{ f_{n}(x) \} \) is bounded for any rational number \( x \in (a,b) \), contradiction.

8. Take \( \varepsilon > 0 \) and for \( k \in \mathbb{N} \) define
\[
F_{k} = \{ x \in X | \tilde{d}(f_{n}(x), f_{m}(x)) \leq \varepsilon/2 \text{ for all } n,m \geq k \}.
\]
The set \( F_{k} \) is closed. To see this note that \( x \mapsto \tilde{d}(f_{n}(x), f_{m}(x)) \) is continuous as a composition of continuous maps. Thus, if \( (x_{j}) \subset F_{k} \) and \( d(x_{j}, x_{0}) \to 0 \) as \( j \to \infty \), then
\[
d(f_{n}(x_{0}), f_{m}(x_{0})) = \lim_{j \to \infty} d(f_{n}(x_{j}), f_{m}(x_{j})) \leq \varepsilon/2.
\]
Moreover, \( X = \bigcup_{k \geq 1} F_{k} \) since for every \( x \in X \), \( \{ f_{n}(x) \} \) converges. By Baire’s theorem, \( F_{k}^{0} \neq \emptyset \) for some \( k \). Thus, there exists \( y \) and \( r > 0 \) such that \( B(y,r) \subset F_{k} \). Put \( U = B(y,r) \). If \( x \in U \), then for all \( n,m \geq k \),
\[
d(f_{n}(x), f_{m}(x)) \leq \varepsilon/2 < \varepsilon.
\]