Some solutions to Problem Set 5.

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1. (a) Not a contraction, for example, $d(f(0), f(-1)) = e - 1 > d(0, -1) = 1$.
(b) Not a contraction. Indeed if f is a contraction, then $|f(x) - f(y)| \leqslant c|x - y|$ for some $c < 1$ and all $x, y \geq 0$. On the other hand, the mean value theorem implies that for every $y > 0$ there is $z := z_y \in (0, y)$ such that $|f(0) - f(y)| = e^{-z}|y| \leqslant c|y|$ so that $e^{-z} < c$. But if $y \to 0$, then $z \to 0$ and $e^{-z} \to 1$, contradiction.
(c) f is a contraction by the mean value theorem, since $|f'(c)| = |-e^{c-e^c}| \leq e^{-1}$ for all $c \geq 0$.
(d) f is not a contraction since if $|f(x) - f(y)| \leqslant c|x - y|$ with $c < 1$, then, in particular, $|f(\pi/2) - f(y)| = |f(y)| \leqslant c|\pi/2 - y|$. By the mean value theorem, for every $y > \pi/2$, there is $z \in (\pi/2, y)$ so that $|f(y)| = |f(\pi/2) - f(y)| = |\sin(z)| \cdot |y - \pi/2|$. Hence $|\sin(z)| \leqslant c$. But if $y \to \pi/2$, then $z \to \pi/2$ and so $\sin z \to 1$, contradiction.
(e) f is a contraction. First, $\cos : \mathbb{R} \to \mathbb{R}$ satisfies $|\cos x - \cos y| \leqslant |x - y|$ and $\cos(\mathbb{R}) = [-1, 1]$. Considering $x \mapsto \cos x$ on the set $[-1, 1]$, it is a contraction with the constant $\sin 1 < 1$. So $|\cos(\cos x) - \cos(\cos y)| \leqslant (\sin 1) \cdot |\cos x - \cos y| \leqslant (\sin 1) \cdot |x - y|$.

2. f is a contraction with respect to d_1 but not with respect to d_2 and d_∞.
Take $x = (1, 1)$ and $y = (0, 0)$. Then $f(1, 1) = (8/5, 1/5)$ and $f(0, 0) = (0, 0)$. Then, $d_2((1, 1), (0, 0)) = \sqrt{2}$ and

$$d_2((f(1,1), f(0,0)) = d_2((8/5, 1/5), (0,0)) = \sqrt{65/25} = \sqrt{13/10}\cdot\sqrt{2} = \sqrt{13/10}\cdot d_2((1,1),(0,0)).$$

Similarly, $d_\infty((1, 1), (0, 0)) = 1$ and

$$d_\infty((f(1,1), f(0,0)) = d_2((8/5, 1/5), (0,0) = \max\{8/5, 1/5\} = 8/5 = 8/5\cdot d_\infty((1,1),(0,0)).$$

For the d_1 metric we have

$$\begin{aligned} d_1(f(x_1, y_1), f(x_2, y_2)) &= d_1\big(\frac{1}{10}(8x_1 + 8y_1, x_1 + y_1), \frac{1}{10}(8x_2 + 8y_2, x_2 + y_2)\big) \\ &= \frac{8}{10}|(x_1 + y_1) - (x_2 + y_2)| + \frac{1}{10}|(x_1 + y_1) - (x_2 + y_2)| \\ &= \frac{9}{10}|(x_1 - x_2) + (y_1 - y_2)| = \frac{9}{10}d_\infty((x_1, y_1), (x_2, y_2)). \end{aligned}$$

So f is a contraction with respect to the metric d_1 with contraction constant $9/10$.

3. (a) For $x, y \in (0, a]$,

$$|x^2 - y^2| = |x + y| \cdot |x - y| \leqslant [|x| + |y|] \cdot |x - y| \leqslant (2a) \cdot |x - y|$$

so that f is a contraction if $a < 1/2$. Observe that if $x \in (0, a]$ with $a < 1/2$, then clearly $0 \leqslant x^2 \leqslant a^2 < a$ so that $f : (0, a] \to (0, a]$. If $x^2 = x$, then $x = 0$ or $x = 1$. Hence $f : X \to X$ does not have a fixed point in X. Note that (X, d) is not complete.
(b) Clearly, $x + \dfrac{1}{x} \geq 1$ for $x \geq 1$ so that $f : X \to X$. Observe next that

$$\left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right) = \left(1 - \frac{1}{xy}\right) \cdot (x - y)$$

so that

$$|f(x) - f(y)| = \left(1 - \frac{1}{xy}\right) \cdot |x - y| < |x - y|$$

since $1 - 1/(xy) < 1$ for all $x, y \geq 1$. Assume that f has a fixed point x. Then $x + \dfrac{1}{x} = x$, so $\dfrac{1}{x} = 0$, contradiction. Note that f is not a contraction since otherwise $1 - 1/(xy) \leqslant \alpha$ for some constant $\alpha < 1$ and all $x, y \geq 1$. Taking $y = 1$ and x large we get a contradiction.

4. The space $(\overline{B}(x_0, r_0), d)$ is complete (as it is a closed subspace of a complete metric space) so it suffices to show that $f : (\overline{B}(x_0, r_0)) \to \overline{B}(x_0, r_0)$. If $x \in \overline{B}(x_0, r_0)$, then

$$\begin{aligned} d(f(x), x_0) &\leqslant d(f(x), f(x_0)) + d(f(x_0), x_0) \leqslant \alpha d(x, x_0) + r_0(1-\alpha) \\ &\leqslant \alpha r_0 + (1-\alpha) r_0 = r_0. \end{aligned}$$

Hence $f : \overline{B}(x_0, r_0) \to \overline{B}(x_0, r_0)$ is a contraction, and f has a unique fixed point in $\overline{B}(x_0, r_0)$ by the Banach fixed point theorem.

5 (a) Observe that $[f(x)]^3 - e^x[f(x)]^2 + \dfrac{f(x)}{2} = e^x$ if and only if

$$e^x + \frac{1}{2}\frac{f(x)}{1+f(x)^2} = f(x) \text{ for all } x \in [0,1].$$

Thus, f is the fixed point of $T : C[0,1] \to C[0,1]$ where

$$(Tf)(x) = e^x + \frac{1}{2}\frac{f(x)}{1+f(x)^2}.$$

By the mean value theorem applied to the function $h(t) = \dfrac{t}{1+t^2}$, there exists $c \in (a,b)$ such that

$$\left|\frac{a}{1+a^2} - \frac{b}{1+b^2}\right| = \left|\frac{1-c^2}{(1+c^2)^2}\right| \cdot |a-b| \leqslant |a-b|.$$

Thus, if $f, g \in C[0,1]$, then

$$|(Tf)(x) - (Tg)(x)| = \frac{1}{2}\left|\frac{f(x)}{1+f(x)^2} - \frac{g(x)}{1+g(x)^2}\right| \leqslant \frac{1}{2} \cdot |f(x) - g(x)|$$

for all $x \in [0,1]$. Consequently, if d_∞ is the supremum metric,

$$d_\infty((Tf), (Tg)) \leqslant \frac{1}{2} \cdot d_\infty(f,g)$$

so that $T : C[0,1] \to C[0,1]$ is a contraction. Since $(C[0,1], d_\infty)$ is complete, there is exactly one f such that $Tf = f$.

(b) For $t \in [0,a]$ we have

$$|(Tf)(t) - (Tg)(t)| \leqslant \int_0^t |f(s) - g(s)| ds \leqslant a \cdot d_\infty(f,g).$$

Hence $d_\infty(Tf, Tg) \leqslant a \cdot d_\infty(f,g)$ and so T is a contraction. Hence there exists exactly one f such that $Ff = f$. That is,

$$f(t) = \sin t + \int_0^t f(s) ds, \quad t \in [0,a].$$

Evaluating at $t = 0$ we get $f(0) = 0$. Differentiating both sides we get $f'(t) = \cos t + f(t)$ for $t \in [0,a]$. Multiplying by e^{-t} this is equivalent to $[e^{-t}f(t)]' = e^{-t}\cos t$. Integrating from 0 to s we get $e^{-s}f(s) = \int_0^s e^{-t}\cos t\, dt = 1/2 + e^{-s}[\sin s - \cos s]/2$ which implies that $f(t) = e^t/2 + [\sin t - \cos t]/2$.

(c) Define $T : C[0,\pi] \to C[0,\pi]$ by

$$(Tf)(t) = \frac{1}{3} \cdot \int_0^t \sin(t-s) \cdot f(s)\, ds.$$

Now for $0 \leq s \leq t \leq \pi$ we have $\sin(t-s) \geq 0$, hence

$$|(Tf)(t) - (Tg)(t)| \leqslant \frac{1}{3}\int_0^t \sin(t-s)|f(s) - g(s)|\, ds \leqslant \frac{1}{3}\int_0^\pi \sin(t-s) d(f,g)\, ds = \frac{2}{3} d(f,g),$$

where d is the sup metric. Thus T is a contraction with contraction constant $2/3$. So there is exactly one f in $C[0,\pi]$ such that $3f(t) = \int_0^t \sin(t-s) f(s) ds$ for all $t \in [0,\pi]$. Since the zero function solves this equation, $f \equiv 0$.

(d) The equation is equivalent to

$$g(x) - \int_0^1 e^{x-y-1} f(y)dy = f(x) \quad \text{for all } x \in [0,1].$$

So setting

$$(Tf)(x) = g(x) - \int_0^1 e^{x-y-1} f(y)dy \quad \text{for } x \in [0,1],$$

we have to show that there is a fixed point of T. We equip $C[0,1]$ with the metric $d(f,h) = \sup\{e^{-x}|f(x)-h(x)| : x \in [0,1]\}$. Then $(C[0,1], d)$ is complete. Since

$$\begin{aligned}|Tf(x) - Th(x)| &\leqslant \int_0^1 e^{x-y-1}|f(y)-h(y)|\, dy = e^x \cdot e^{-1} \int_0^1 e^{-y}|f(y)-h(y)|dy \\ &\leqslant e^x \cdot e^{-1} \cdot d(f,h)\end{aligned}$$

we conclude $d(Tf, Th) \leqslant e^{-1} d(f,h)$. By Banach fixed point theorem there is exactly one f such that $Tf = f$.

6. For $k \geq 1$ let $U_k = \{x \in \mathbb{R}^n |\ f_k(x) \neq 0\} = f_k^{-1}(\mathbb{R} \setminus \{0\})$. Since $\mathbb{R} \setminus \{0\}$ is open and f_k is continuous (a linear map from a finite dimensional vector space to another vector space), we conclude that U_k is open. We claim that every U_k is dense. If not, then for some k, there exists $B(a,r) \subset \mathbb{R}^n$ such that $B(a,r) \cap U_k = \emptyset$. Hence, for every $x \in B(a,r)$, $f_k(x) = 0$. Take any $x \in \mathbb{R}^n$ not equal to a. Then $y := \frac{r(x-a)}{2\|x-a\|} \in B(a,r)$ and so $f(y) = 0$. But since f_k is linear,

$$0 = f(y) = \frac{r}{\|x-a\|} f(x) - \frac{r}{\|x-a\|} f(a) = \frac{r}{\|x-a\|} f(x)$$

so that $f_k(x) = 0$. Hence $f_k = 0$ contradicting the assumption. Hence U_k is dense and open. In view of the Baire's theorem, $\bigcap_{k \geq 1} U_k$ is dense, in particular, non-empty. Take any $x \in \bigcap_{k \geq 1} U_k$ and the result follows.

7. Arguing by contradiction assume that $\{f_n(x)\}$ is bounded for every irrational x. For $k \geq 1$ define

$$F_k = \{x \in \mathbb{R} \mid |f_n(x)| \leqslant k \text{ for all } n \in \mathbb{N}\} = \bigcap_{n \in \mathbb{N}} f_n^{-1}([-k,k]).$$

The set $f_n^{-1}([-k,k])$ is closed since f_n is continuous, and so F_k is closed. Moreover, $\mathbb{Q}^c \subset \bigcup_{k \in \mathbb{N}} F_k$ since, by assumption, $\{f_n(x)\}$ is bounded for every irrational x. So

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} F_k \cup \bigcup_{r \in \mathbb{Q}} \{r\},$$

that is, $\mathbb{R}$ is a countable union of closed sets since $\mathbb{Q}$ is countable. Every singleton $\{r\}$ is nowhere dense. Consequently, in view of Baire theorem, some F_k has nonempty interior. Say $F_k^\circ \neq \emptyset$. So $(a,b) \subset F_k$, and $\{f_n(x)\}$ is bounded for any $x \in (a,b)$. In particular, $\{f_n(x)\}$ is bounded for any rational number $x \in (a,b)$, contradiction.

8. Take $\varepsilon > 0$ and for $k \in \mathbb{N}$ define

$$F_k = \{x \in X |\ \widetilde{d}(f_n(x), f_m(x)) \leqslant \varepsilon/2 \text{ for all } n, m \geq k\}.$$

The set F_k is closed. To see this note that $x \mapsto \overline{d}(f_n(x), f_m(x))$ is continuous as a composition of continuous maps. Thus, if $(x_j) \subset F_k$ and $d(x_j, x_0) \to 0$ as $j \to \infty$, then

$$d(f_n(x_0), f_m(x_0)) = \lim_{j \to \infty} d(f_n(x_j), f_m(x_j)) \leqslant \varepsilon/2.$$

Moreover, $X = \bigcup_{k \geq 1} F_k$ since for every $x \in X$, $\{f_n(x)\}$ converges. By Baire's theorem, $F_k^\circ \neq \emptyset$ for some k. Thus, there exists y and $r > 0$ such that $B(y,r) \subset F_k$. Put $U = B(y,r)$. If $x \in U$, then for all $n, m \geq k$,

$$d(f_n(x), f_m(x)) \leqslant \varepsilon/2 < \varepsilon.$$