Some solutions to Problem Set 6.

1. The space \((\mathbb{R}, d_1)\) is incomplete. Indeed, let \(x_n = n\). Then \(d_1(x_n, x_m) = |\arctan n - \arctan m| \to 0\) as \(n, m \to \infty\). So \((x_n)\) is Cauchy sequence. However, if \(x \in \mathbb{R}\) and \(d_1(x, x) \to 0\), then \(|\arctan n - \arctan x| \to 0\) and since \(\arctan n \to \pi/2\), it follows that \(\arctan x = \pi/2\) which is impossible. The completion \((\mathbb{R}, \tilde{d}_1)\) may be defined as follows, let \(\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}\) and \(\tilde{d}_1(x, y) = d_1(x, y)\) if \(x, y \in \mathbb{R}\), \(\tilde{d}_1(x, -\infty) = \tilde{d}_1(-\infty, x) = \arctan x + \pi/2\) for \(x \in \mathbb{R}\), \(\tilde{d}_1(x, \infty) = \tilde{d}_1(\infty, x) = \pi/2 - \arctan x\) for \(x \in \mathbb{R}\), \(\tilde{d}_1(\infty, -\infty) = \tilde{d}_1(-\infty, \infty) = \pi\), and \(\tilde{d}_1(\pm \infty, \pm \infty) = 0\). Then \((\mathbb{R}, \tilde{d}_1)\) is complete and the set \(\mathbb{R}\) is dense in \((\mathbb{R}, \tilde{d}_1)\). Another completion is given by \([\mathbb{R}, d] = \mathbb{R}\). The map \(\arctan : \mathbb{R} \to [-\pi/2, \pi/2] \subset [-\pi/2, \pi/2]\) is an isometry, and \(\arctan(\mathbb{R}) = (-\pi/2, \pi/2)\) is dense in \([-\pi/2, \pi/2]\).

The space \((\mathbb{R}, d_2)\) is complete. Indeed, \((x_n)\) is Cauchy with respect to \(d_2\), then \((x_n^3)\) is Cauchy with respect to the standard metric and so there is \(y \in \mathbb{R}\) such that \(|x_n^3 - y| \to 0\). Let \(x = \sqrt[3]{y}\). Then \(d_2(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \to 0\).

2. (a) \(A\) is not compact. Indeed, let \(x\) be an irrational number in \([0, 1]\). Then there is a sequence \((x_n)\) in \(A\) with \(x_n \to x\). So \((x_n)\) doesn’t have a subsequence converging to a point in \(A\). (b) \(B\) is compact in \(\mathbb{R}^2\) since it is closed and bounded. (c) \(C\) is not compact since it is not closed. (d) \(D\) is compact in \(\mathbb{R}^2\) since it is closed and bounded. (e) \(E\) is not compact since it is unbounded.

3. Let \((x_n)\) be any sequence in \(\bigcup_{i=1}^k A_i\). Then there is \(A_i\) and a strictly increasing sequence \(n_1 < n_2 < \cdots\) of positive integers such that \(y_k = x_{n_k} \in A_i\). Since \(A_i\) is compact, there exists a subsequence \((y_k)\) of \((x_k)\) such that \((y_k)\) converges in \(A_i\). So the subsequence \((x_{n_k})\) of \((x_n)\) converges in \(A_i\).

4. We prove this for \(k = 2\); then the general result follows by induction. Let \(\{(x_n, y_n)\}\) be any sequence in \(A_1 \times A_2\). Since \(A_1\) is compact, there exists a subsequence \((x_{n_k})\) of \((x_n)\) converging to some point \(x\) in \(A_1\). Since \(A_2\) is compact, the sequence \((y_{n_k})\) converges to some point \(y\) in \(A_2\). Then the subsequence \(\{(x_{n_k}, y_{n_k})\}\) of \(\{(x_n, y_n)\}\) converges to \((x, y)\) in the product metric \(d\). Hence \((A_1 \times A_2, d)\) is compact.

5. (a) The function \(f : A \to \mathbb{R}\) defined by \(f(a) = d(x, a)\) is continuous. Since \(A\) is compact, there is \(a \in A\) such that \(f(a) \leq f(b)\) for all \(b \in A\) that is, \(d(x, a) \leq d(x, b)\) for all \(b \in A\) implying that \(d(x, a) \leq d(x, A)\). Since also \(d(x, A) \leq d(x, a)\), we have \(d(x, a) = d(x, A)\).

(b) Arguing by contradiction, assume that for every \(\varepsilon > 0\) there is \(x \not\in U\) such that \(d(x, A) < \varepsilon\). In particular, taking \(\varepsilon = 1/n\) we find a sequence \((x_n)\) such that \(x_n \in X \setminus U\) and \(d(x_n, A) < 1/n\). By (a), there is \(a_n \in A\) such that \(d(x_n, a_n) = d(x_n, A) < 1/n\). Since \(A\) is compact, there is a subsequence \((a_{n_k})\) converging to a point \(a \in A\). Since \(d(x_{n_k}, a_{n_k}) \to 0\), we have that also \(x_{n_k} \to a\) and since \(X \setminus U\) is closed, \(a \in X \setminus U \subset X \setminus A\), contradiction.
(c) Arguing by contradiction, assume that \(d(A, B) = 0\). Then for every \(n\), there \(a_n \in A\) and \(b_n \in B\) such that \(d(a_n, b_n) < 1/n\). Since \(A\) is compact, there is subsequence \(\{a_{n_k}\}\) converging to \(a \in A\). From \(d(a_{n_k}, b_{n_k}) < 1/n_k \to 0\) it follows that \(b_{n_k} \to a\) so that \(a\) is an adherent point of \(B\). Since \(B\) is closed, \(a \in B\). Hence \(a \in A \cap B\), contradiction.

Alternative approach: In (b), if \(X \neq U\) then \(X \setminus U\) is closed and non-empty, and the function \(g : A \to \mathbb{R}\) defined by \(g(a) = d(a, X \setminus U)\) is continuous, so attains its minimum on the compact set \(A\). Since \(A \cap (X \setminus U) = \emptyset\), it follows that \(d(A, X \setminus U) = \min g = \varepsilon > 0\) and \(\{x \in X : d(x, A) < \varepsilon\} \subset U\). In (c), if \(B\) is closed and \(A \cap B = \emptyset\) then \(U = X \setminus B\) is open and \(A \subset U\). Now the result follows from part (b).

6. Consider an u.s.c function \(f : X \to \mathbb{R}\). For every \(n\) define \(U_n = \{x \in X | f(x) < n\}\). So \(U_n\) is open. Moreover, \(X = \bigcup_{n \in \mathbb{N}} U_n\). Since \(X\) is compact, there is \(N\) such that \(X = \bigcup_{n \leq N} U_n = U_N\). Hence the set \(\{f(x) | x \in X\}\) is bounded from above so that \(a = \sup \{f(x) | x \in X\} < \infty\). Suppose that there is no \(x\) such that \(f(x) = a\). Then \(f(x) < a\) for all \(x \in X\). For every \(n \in \mathbb{N}\), let \(V_n = \{x \in X | f(x) < a - 1/n\}\). Each \(V_n\) is open and \(X = \bigcup_{n \in \mathbb{N}} V_n\). Indeed, if \(x \in X\) then since \(f(x) < a\) there is \(n\) such that \(f(x) < a - 1/n\), i.e., \(x \in V_n\). Since \(X\) is compact, there is \(N\) such that \(X = \bigcup_{n \leq N} V_n = V_N\). This means that \(f(x) < a - 1/N\) for all \(x \in X\) which implies \(a = \sup \{f(x) | x \in X\} \leq a - 1/N\), contradiction. The proof for a l.s.c function is similar.

7. If \(f(x) = x, f(y) = y\) and \(x \neq y\), then \(d(x, y) = d(f(x), f(y)) < d(x, y)\), contradiction. So \(f\) has at most one fixed point. To see that \(f\) has a fixed point consider the function \(g : X \to \mathbb{R}\) defined by \(g(x) = d(x, f(x))\). Then it suffices to show that from some \(x\), \(g(x) = 0\). Since \(X\) is compact and \(g\) is continuous, \(g\) attains a minimum, so there is \(x_0\) such that \(g(x_0) \leq g(x)\) for all \(x \in X\). If \(g(x_0) > 0\), i.e., \(x_0 \neq f(x_0)\), then \(g(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = g(x_0)\) contradicting the minimality of \(g(x_0)\).

8. (a) Let \(x = \{x_n\}, y = \{y_n\}\) and \(z = \{z_n\} \in X^*\). Since \(d(x_n, x_n) = 0, x \sim x,\) and since \(d(x_n, y_n) = d(y_n, x_n)\) we conclude that \(x \sim y\) implies \(y \sim x\). Finally, if \(x \sim y\) and \(y \sim z\), then \(d(x_n, y_n) \to 0\) and \(d(y_n, z_n) \to 0\) so that \(d(x_n, z_n) \to 0\) since \(d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \to 0\). Hence \(x \sim z\), and \(\sim\) is indeed an equivalence relation.

(b) Let \(x = \{x_n\}\) and \(y = \{y_n\} \in X^*\). Since
\[
d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)
\]
we have \(d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n)\). Writing the same inequality but with \(m\) and \(n\) interchanged, we get
\[
|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m).
\]
It follows that \(\{d(x_n, y_n)\}\) is Cauchy and since \(\mathbb{R}\) with the usual metric is complete, the sequence \(\{d(x_n, y_n)\}\) converges.
Next, we show that if \( x'_n \in [x] \) and \( y'_n \in [y] \), then
\[
(1) \quad \lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x'_n, y'_n).
\]
Since
\[
d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n),
\]
and \( d(x_n, x'_n) \to 0, d(y'_n, y_n) \to 0 \) we get
\[
(2) \quad \lim_{n \to \infty} d(x_n, y_n) \leq \lim_{n \to \infty} d(x'_n, y'_n).
\]
Interchanging \( x_n \) with \( x'_n \) and \( y_n \) with \( y'_n \), we get
\[
\lim_{n \to \infty} d(x'_n, y'_n) \leq \lim_{n \to \infty} d(x_n, y_n)
\]
which together with (2) implies (1).
(c) \( D \) is a metric. Indeed, \( D([x], [y]) \geq 0 \). If \( D([x], [y]) = 0 \) then \( \lim_{n \to \infty} d(x_n, y_n) = 0 \). This means that \( x \sim y \), and so \([x] = [y]\). Symmetry of \( D \) follows from the fact that \( d(x_n, y_n) = d(y_n, x_n) \). To prove the triangle inequality, take \([x], [y] \) and \([z] \in \bar{X} \). Since the limits \( \lim_{n \to \infty} d(x_n, z_n), \lim_{n \to \infty} d(x_n, y_n) \) and \( \lim_{n \to \infty} d(y_n, z_n) \) exist, and \( d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \), we have
\[
D([x], [z]) = \lim_{n \to \infty} d(x_n, z_n) \leq \lim_{n \to \infty} d(x_n, y_n) + \lim_{n \to \infty} d(y_n, z_n)
\]
\[
= D([x], [y]) + D([y], [z]).
\]
Next we will show that \((\bar{X}, D)\) is complete. Take a Cauchy sequence \( \{x^n\} \) in \((\bar{X}, D)\). Here \( x^n = \{x^n_1, x^n_2, x^n_3, \ldots\} \) denotes a Cauchy sequence in \((X, d)\). We have to show that there exists \([x] \in \bar{X} \) such that \( D([x^n], [x]) \to 0 \). Since \( x^n = \{x^n_k\} \) is Cauchy in \((X, d)\), for every \( n \in \mathbb{N} \) there exists \( k_n \in \mathbb{N} \) such that
\[
(3) \quad d(x^n_{m}, x^n_{k_n}) < \frac{1}{n} \quad \text{for all } m \geq k_n.
\]
Consider the sequence
\[
x = \{x_n\} = \{x^1_1, x^2_1, x^3_1, \ldots\}.
\]
We claim that \( x \) is Cauchy in \((X, d)\). Indeed, denoting by \( x^n_{k_n} \) the constant sequence \( \{x^n_{k_n}, x^n_{k_n}, x^n_{k_n}, \ldots\} \), we have
\[
(4) \quad D([x^n], [x^n_{k_n}]) = \lim_{j \to \infty} d(x^n_j, x^n_{k_n}) \leq \frac{1}{n}.
\]
Let \( \varepsilon > 0 \) and choose \( N \in \mathbb{N} \) such that \( 1/N < \varepsilon/3 \) and
\[
D([x^n], [x^m]) < \varepsilon/3 \quad \text{for } n, m \geq N.
\]
Then,
\[
d(x^n_{k_n}, x^m_{k_m}) = D([x^n_{k_n}], [x^m_{k_m}])
\]
\[
\leq D([x^n_{k_n}], [x^n]) + D([x^n], [x^m]) + D([x^m], [x^m_{k_m}])
\]
\[
\leq \frac{1}{n} + D([x^n], [x^m]) + \frac{1}{m} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]
for all \( n, m \geq N \). Next we will show that \( D([x^n], [x]) \to 0 \). In view of (4),
\[
(5) \quad D([x], [x^n]) \leq D([x], [x^n_{k_n}]) + D([x^n_{k_n}], [x^n]) \leq D([x], [x^n_{k_n}]) + \frac{1}{n}.
\]
Take $\varepsilon > 0$. Since, in view of (5), the sequence $\{x^n_{k_n}\}$ is Cauchy, there exists $k \in \mathbb{N}$ such that $1/k < \varepsilon/2$ and

$$d(x^n_{k_n}, x^m_{k_m}) < \varepsilon/2 \quad \text{for all } n, m \geq k. \tag{7}$$

Fixing $n \geq k$ and taking a limit as $m \to \infty$ in (7), we get

$$D([x^n_{k_n}], [x]) = \lim_{m \to \infty} d(x^n_{k_n}, x^m_{k_m}) \leq \varepsilon/2 \tag{8}$$

Combining (6) with (8),

$$D([x], [x^n]) \leq D([x], [x^n_{k_n}]) + \frac{1}{n} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

for all $n \geq k$. Consequently, $D([x^n], [x]) \to 0$, and $(\tilde{X}, D)$ is complete.

(d) If $x = (x, x, x, \ldots)$ and $y = (y, y, y, \ldots)$ are constant sequences in $X$, then $\varphi(x) = [x], \varphi(y) = [y]$ so that

$$D([x], [y]) = \lim_{n \to \infty} d(x, y) = d(x, y).$$

Hence $\varphi : X \to \varphi(X)$ is an isometry.

(e) It suffices to show that for any $[x] \in \tilde{X}$ there exists a sequence $[x^n] \in \varphi(X)$ such that $D([x^n], [x]) \to 0$. Let $[x] \in \tilde{X}$ with $x = (x_1, x_2, x_3 \ldots)$. Denote by $x^n$ the constant sequence $\{x_n, x_n, x_n, \ldots\}$. Then $\varphi(x_n) = [x^n]$, and we claim that $D([x^n], [x]) \to 0$. Since $x$ is Cauchy, for given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon/2 \quad \text{for } n, m \geq k.$$

Fixing $n \geq k$ and taking a limit as $m \to \infty$, we get

$$\lim_{m \to \infty} d(x_n, x_m) \leq \varepsilon/2.$$

So, for $n \geq k$,

$$D([x^n], [x]) = \lim_{m \to \infty} d(x_n, x_m) \leq \varepsilon/2 < \varepsilon.$$

Thus, $D([x^n], [x]) \to 0$ as required.