Some solutions to Problem Set 6.

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1. The space $(\mathbb{R}, d_1)$ is incomplete. Indeed, let $x_n = n$. Then $d_1(x_n, x_m) = |\arctan n - \arctan m| \to 0$ as $n, m \to \infty$. So (x_n) is Cauchy sequence. However, if $x \in \mathbb{R}$ and $d_1(x_n, x) \to 0$, then $|\arctan n - \arctan x| \to 0$ and since $\arctan n \to \pi/2$, it follows that $\arctan x = \pi/2$ which is impossible. The completion $(\widehat{\mathbb{R}}, \widehat{d_1})$ may be defined as follows, let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\widehat{d_1}(x, y) = d_1(x, y)$ if $x, y \in \mathbb{R}$, $\widehat{d_1}(x, -\infty) = \widehat{d_1}(-\infty, x) = \arctan x + \pi/2$ for $x \in \mathbb{R}$, $\widehat{d_1}(x, \infty) = \widehat{d_1}(\infty, x) = \pi/2 - \arctan x$ for $x \in \mathbb{R}$, $\widehat{d_1}(\infty, -\infty) = \widehat{d_1}(-\infty, \infty) = \pi$, and $\widehat{d_1}(\pm\infty, \pm\infty) = 0$. Then $(\widehat{\mathbb{R}}, \widehat{d_1})$ is complete and the set $\mathbb{R}$ is dense in $(\widehat{\mathbb{R}}, \widehat{d_1})$. Another completion is given by $([-\pi/2, \pi/2], d)$ where d is the standard metric. The map $\arctan : \mathbb{R} \to (-\pi/2, \pi/2) \subset [-\pi/2, \pi/2]$ is an isometry, and $\arctan(\mathbb{R}) = (-\pi/2, \pi/2)$ is dense in $[-\pi/2, \pi/2]$.

The space $(\mathbb{R}, d_2)$ is complete. Indeed, (x_n) is Cauchy with respect to d_2, then (x_n^3) is Cauchy with respect to the standard metric and so there is $y \in \mathbb{R}$ such that $|x_n^3 - y| \to 0$. Let $x = \sqrt[3]{y}$. Then $d_2(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \to 0$.

2. (a) A is not compact. Indeed, let x be an irrational number in $[0, 1]$. Then there is a sequence (x_n) in A with x_n such that $|x_n - x| \to 0$. So (x_n) doesn't have a subsequence converging to a point in A. (b) B is compact in $\mathbb{R}^2$ since it is closed and bounded. (c) C is not compact since it is not closed. (d) D is compact in $\mathbb{R}^2$ since it is closed and bounded. (e) E is not compact since it is unbounded.

3. Let (x_n) be any sequence in $\bigcup_{i=1}^{k} A_i$. Then there is A_i and a strictly increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that $y_k = x_{n_k} \in A_i$. Since A_i is compact, there exists a subsequence (y_{k_i}) of (y_k) such that (y_{k_i}) converges in A_i. So the subsequence $(x_{n_{k_i}})$ of (x_n) converges in A_i.

4. We prove this for $k = 2$; then the general result follows by induction. Let $\{(x_n, y_n)\}$ be any sequence in $A_1 \times A_2$. Since A_1 is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point x in A_1. Since A_2 is compact, then the sequence $\{y_{n_k}\}$ has a subsequence $\{y_{n_{k_l}}\}$ converging to some point $y \in A_2$. Then the subsequence $\{(x_{n_{k_l}}, y_{n_{k_l}})\}$ of $\{(x_n, y_n)\}$ converges to (x, y) in the product metric d. Hence $(A_1 \times A_2, d)$ is compact.

5. (a) The function $f : A \to \mathbb{R}$ defined by $f(a) = d(x, a)$ is continuous. Since A is compact, there is $a \in A$ such that $f(a) \leqslant f(b)$ for all $b \in A$. That is, $d(x, a) \leqslant d(x, b)$ for all $b \in A$ implying that $d(x, a) \leqslant d(x, A)$. Since also $d(x, A) \leqslant d(x, a)$, we have $d(x, a) = d(x, A)$.

(b) Arguing by contradiction, assume that for every $\varepsilon > 0$ there is $x \notin U$ such that $d(x, A) < \varepsilon$. In particular, taking $\varepsilon = 1/n$ we find a sequence $\{x_n\}$ such that $x_n \in X \setminus U$ and $d(x_n, A) < 1/n$. By (a), there is $a_n \in A$ such that $d(x_n, a_n) = d(x_n, A) < 1/n$. Since A is compact, there is a subsequence $\{a_{n_k}\}$ converging to a point $a \in A$. Since $d(x_{n_k}, a_{n_k}) \to 0$, we have that also $x_{n_k} \to a$ and since $X \setminus U$ is closed, $a \in X \setminus U \subset X \setminus A$, contradiction.

(c) Arguing by contradiction, assume that $d(A,B)=0$. Then for every n, there $a_n \in A$ and $b_n \in B$ such that $d(a_n,b_n) < 1/n$. Since A is compact, there is subsequence $\{a_{n_k}\}$ converging to $a \in A$. From $d(a_{n_k},b_{n_k}) < 1/n_k \to 0$ it follows that $b_{n_k} \to a$ so that a is an adherent point of B. Since B is closed, $a \in B$. Hence $a \in A \cap B$, contradiction.

Alternative approach: In (b), if $X \neq U$ then $X \setminus U$ is closed and non-empty, and the function $g : A \to \mathbb{R}$ defined by $g(a) = d(a, X \setminus U)$ is continuous, so attains its minimum on the compact set A. Since $A \cap (X \setminus U) = \emptyset$, it follows that $d(A, X \setminus U) = \min g = \varepsilon > 0$ and $\{x \in X : d(x,A) < \varepsilon\} \subset U$. In (c), if B is closed and $A \cap B = \emptyset$ then $U = X \setminus B$ is open and $A \subset U$. Now the result follows from part (b).

6. Consider an u.s.c function $f : X \to \mathbb{R}$. For every n define $U_n = \{x \in X | f(x) < n\}$. So U_n is open. Moreover, $X = \bigcup_{n\in\mathbb{N}} U_n$. Since X is compact, there is N such that $X = \bigcup_{n\leqslant N} U_n = U_N$. Hence the set $\{f(x)|\ x \in X\}$ is bounded from above so that $a = \sup\{f(x)|\ x \in X\} < \infty$. Suppose that there is no x such that $f(x) = a$. Then $f(x) < a$ for all $x \in X$. For every $n \in \mathbb{N}$, let $V_n = \{x \in X | f(x) < a - 1/n\}$. Each V_n is open and $X = \bigcup_{n\in\mathbb{N}} V_n$. Indeed, if $x \in X$ then since $f(x) < a$ there is n such that $f(x) < a - 1/n$, i.e., $x \in V_n$. Since X is compact, there is N such that $X = \bigcup_{n\leqslant N} V_n = V_N$. This means that $f(x) < a - 1/N$ for all $x \in X$ which implies $a = \sup\{f(x)|\ x \in X\} \leqslant a - 1/N$, contradiction. The proof for a l.s.c function is similar.

7. If $f(x) = x$, $f(y) = y$ and $x \neq y$, then $d(x,y) = d(f(x), f(y)) < d(x,y)$, contradiction. So f has at most one fixed point. To see that f has a fixed point consider the function $g : X \to \mathbb{R}$ defined by $g(x) = d(x, f(x))$. Then it suffices to show that from some x, $g(x) = 0$. Since X is compact and g is continous, g attains a minimum, so there is x_0 such that $g(x_0) \leqslant g(x)$ for all $x \in X$. If $g(x_0) > 0$, i.e., $x_0 \neq f(x_0)$, then $g(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0) = g(x_0)$ contradicting the minimality of $g(x_0)$.

8. (a) Let $\mathbf{x} = \{x_n\}, \mathbf{y} = \{y_n\}$ and $\mathbf{z} = \{z_n\} \in X^*$. Since $d(x_n,x_n) = 0$, $\mathbf{x} \sim \mathbf{x}$, and since $d(x_n,y_n) = d(y_n,x_n)$ we conclude that $\mathbf{x} \sim \mathbf{y}$ implies $\mathbf{y} \sim \mathbf{x}$. Finally, if $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$, then $d(x_n,y_n) \to 0$ and $d(y_n,z_n) \to 0$ so that $d(x_n,z_n) \to 0$ since $d(x_n,z_n) \leqslant d(x_n,y_n) + d(y_n,z_n) \to 0$. Hence $\mathbf{x} \sim \mathbf{z}$, and $\sim$ is indeed an equivalence relation.
(b) Let $\mathbf{x} = \{x_n\}$ and $\mathbf{y} = \{y_n\} \in X^*$. Since

$$d(x_n,y_n) \leqslant d(x_n,x_m) + d(x_m,y_m) + d(y_m,y_n)$$

we have $d(x_n,y_n) - d(x_m,y_m) \leqslant d(x_n,x_m) + d(y_m,y_n)$. Writing the same inequality but with m and n interchanged, we get

$$|d(x_n,y_n) - d(x_m,y_m)| \leqslant d(x_n,x_m) + d(y_n,y_m).$$

It follows that $\{d(x_n,y_n)\}$ is Cauchy and since $\mathbb{R}$ with the usual metric is complete, the sequence $\{d(x_n,y_n)\}$ converges.

Next, we show that if $\{x'_n\} \in [\mathbf{x}]$ and $\{y'_n\} \in [\mathbf{y}]$, then

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n). \tag{1}$$

Since

$$d(x_n, y_n) \leqslant d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n),$$

and $d(x_n, x'_n) \to 0$, $d(y'_n, y_n) \to 0$ we get

$$\lim_{n\to\infty} d(x_n, y_n) \leqslant \lim_{n\to\infty} d(x'_n, y'_n). \tag{2}$$

Interchanging x_n with x'_n and y_n with y'_n, we get

$$\lim_{n\to\infty} d(x'_n, y'_n) \leqslant \lim_{n\to\infty} d(x_n, y_n)$$

which together with (2) implies (1).

(c) D is a metric. Indeed, $D([\mathbf{x}], [\mathbf{y}]) \geq 0$. If $D([\mathbf{x}], [\mathbf{y}]) = 0$ then $\lim_{n\to\infty} d(x_n, y_n) = 0$. This means that $\mathbf{x} \sim \mathbf{y}$, and so $[\mathbf{x}] = [\mathbf{y}]$. Symmetry of D follows from the fact that $d(x_n, y_n) = d(y_n, x_n)$. To prove the triangle inequality, take $[\mathbf{x}]$, $[\mathbf{y}]$ and $[\mathbf{z}] \in \widetilde{X}$. Since the limits $\lim_{n\to\infty} d(x_n, z_n)$, $\lim_{n\to\infty} d(x_n, y_n)$ and $\lim_{n\to\infty} d(y_n, z_n)$ exist, and $d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n)$, we have

$$\begin{aligned} D([\mathbf{x}], [\mathbf{z}]) &= \lim_{n\to\infty} d(x_n, z_n) \leqslant \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &= D([\mathbf{x}], [\mathbf{y}]) + D([\mathbf{y}], [\mathbf{z}]). \end{aligned}$$

Next we will show that $(\widetilde{X}, D)$ is complete. Take a Cauchy sequence $\{[\mathbf{x}^n]\}$ in $(\widetilde{X}, D)$. Here $\mathbf{x}^n = \{x^n_k\} = \{x^n_1, x^n_2, x^n_3, \ldots\}$ denotes a Cauchy sequence in (X, d). We have to show that there exists $[\mathbf{x}] \in \widetilde{X}$ such that $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$. Since $\mathbf{x}^n = \{x^n_k\}$ is Cauchy in (X, d), for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that

$$d(x^n_m, x^n_{k_n}) < \frac{1}{n} \quad \text{for all } m \geq k_n. \tag{3}$$

Consider the sequence

$$\mathbf{x} = \{x_n\} = \{x^1_{k_1}, x^2_{k_2}, x^3_{k_3}, \ldots\}.$$

We claim that $\mathbf{x}$ is Cauchy in (X, d). Indeed, denoting by $\mathbf{x}^n_{k_n}$ the constant sequence $\{x^n_{k_n}, x^n_{k_n}, x^n_{k_n}, \ldots\}$, we have

$$D([\mathbf{x}^n], [\mathbf{x}^n_{k_n}]) = \lim_{j\to\infty} d(x^n_j, x^n_{k_n}) \leqslant \frac{1}{n}. \tag{4}$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \varepsilon/3$ and

$$D([\mathbf{x}^n], [\mathbf{x}^m]) < \varepsilon/3 \quad \text{for } n, m \geq N.$$

Then,

$$\begin{aligned} d(x^n_{k_n}, x^m_{k_m}) &= D([\mathbf{x}^n_{k_n}], [\mathbf{x}^m_{k_m}]) \\ &\leqslant D([\mathbf{x}^n_{k_n}], [\mathbf{x}^n]) + D([\mathbf{x}^n], [\mathbf{x}^m]) + D([\mathbf{x}^m], [\mathbf{x}^m_{k_m}]) \\ &\leqslant \frac{1}{n} + D([\mathbf{x}^n], [\mathbf{x}^m]) + \frac{1}{m} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned} \tag{5}$$

for all $n, m \geq N$. Next we will show that $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$. In view of (4),

$$D([\mathbf{x}], [\mathbf{x}^n]) \leqslant D([\mathbf{x}], [\mathbf{x}^n_{k_n}]) + D([\mathbf{x}^n_{k_n}], [\mathbf{x}^n]) \leqslant D([\mathbf{x}], [\mathbf{x}^n_{k_n}]) + \frac{1}{n}. \tag{6}$$

Take $\varepsilon > 0$. Since, in view of (5), the sequence $\{x^n_{k_n}\}$ is Cauchy, there exists $k \in \mathbb{N}$ such that $1/k < \varepsilon/2$ and

$$d(x^n_{k_n}, x^m_{k_m}) < \varepsilon/2 \quad \text{for all } n, m \geq k. \tag{7}$$

Fixing $n \geq k$ and taking a limit as $m \to \infty$ in (7), we get

$$D([\mathbf{x}^n_{k_n}], [\mathbf{x}]) = \lim_{m\to\infty} d(x^n_{k_n}, x^m_{k_m}) \leqslant \varepsilon/2 \tag{8}$$

Combining (6) with (8),

$$D([\mathbf{x}], [\mathbf{x}^n]) \leqslant D([\mathbf{x}], [\mathbf{x}^n_{k_n}]) + \frac{1}{n} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

for all $n \geq k$. Consequently, $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$, and $(\widetilde{X}, D)$ is complete.

(d) If $\mathbf{x} = (x, x, x, \ldots)$ and $\mathbf{y} = (y, y, y, \ldots)$ are constant sequences in X, then $\varphi(x) = [\mathbf{x}]$, $\varphi(y) = [\mathbf{y}]$ so that

$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Hence $\varphi : X \to \varphi(X)$ is an isometry.

(e) It suffices to show that for any $[\mathbf{x}] \in \widetilde{X}$ there exists a sequence $[\mathbf{x}^n] \in \varphi(X)$ such that $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$. Let $[\mathbf{x}] \in \widetilde{X}$ with $\mathbf{x} = (x_1, x_2, x_3 \ldots)$. Denote by $\mathbf{x}^n$ the constant sequence $\{x_n, x_n, x_n, \ldots\}$. Then $\varphi(x_n) = [\mathbf{x}^n]$, and we claim that $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$. Since $\mathbf{x}$ is Cauchy, for given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon/2 \quad \text{for } n, m \geq k.$$

Fixing $n \geq k$ and taking a limit as $m \to \infty$, we get

$$\lim_{m\to\infty} d(x_n, x_m) \leqslant \varepsilon/2.$$

So, for $n \geq k$,

$$D([\mathbf{x}^n], [\mathbf{x}]) = \lim_{m\to\infty} d(x_n, x_m) \leqslant \varepsilon/2 < \varepsilon.$$

Thus, $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$ as required.