Some solutions to Problem Set 7.

1. None of the spaces is compact. Take the sequence \( \{x_n\} \) with \( x_n = n \). Then \( \{x_n\} \) does not have a convergent subsequence in \( \mathbb{R} \) with the metrics \( d_1 \) and \( d_2 \). (Alternatively: \( (\mathbb{R}, d_1) \) is equivalent to \( \mathbb{R} \) with the usual metric and \( (\mathbb{R}, d_2) \) is isometric to \((-\pi/2, \pi/2) \) with the usual metric.) Also \( d_2(x_n, x_m) = 1 \) for \( n \neq m \), so the sequence does not have a convergent subsequence with the metric \( d_2 \).

2. Let \( \varepsilon > 0 \). For each \( x \in X \), there exists \( \delta_x > 0 \) such that \( d(x, y) < 2\delta_x \) implies \( d(f(x), f(y)) < \frac{1}{2}\varepsilon \). Now \( \{B(x, \delta_x) : x \in X\} \) is an open cover of \( X \) so contains a finite subcover, say \( \{B(x_1, \delta_{x_1}), \ldots, B(x_n, \delta_{x_n})\} \). Let \( \delta = \min(\delta_{x_1}, \ldots, \delta_{x_n}) \). Now let \( y, z \in X \) with \( d(y, z) < \delta \). Then \( y \in B(x_i, \delta_{x_i}) \) for some \( i \) in \( \{1, \ldots, n\} \) so \( d(y, x_i) < \delta_{x_i} \) and \( d(z, x_i) \leq d(z, y) + d(y, x_i) < \delta + \delta_{x_i} \leq \delta_{x_i} + \delta_{x_i} = 2\delta_{x_i} \). Thus \( d(f(z), f(y)) \leq d(f(z), f(x_i)) + d(f(x_i), f(y)) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \). This proves that \( f \) is uniformly continuous.

3. Arguing by contradiction assume that \( \{F_i\}_{i \in I} \) is a family of closed sets having the finite intersection property such that \( \bigcap_{i \in I} F_i = \emptyset \). Let \( U_i = X \setminus F_i \). Then the \( U_i \)'s are open and \( \bigcup_{i \in I} U_i = \bigcup_{i \in I} (X \setminus F_i) = X \setminus \bigcap_{i \in I} F_i = X \). Since \( X \) is compact, there is a finite set \( J \subset I \) such that \( X = \bigcup_{i \in J} U_i \). But then \( \bigcap_{i \in J} F_i = \bigcap_{i \in J} (X \setminus U_i) = X \setminus \bigcup_{i \in J} U_i = \emptyset \), contradicting the fact that \( \bigcap_{i \in J} F_i \neq \emptyset \) for any finite \( J \). Conversely, assume that for every family of closed sets \( \{F_i\}_{i \in I} \) having finite intersection property, we have \( \bigcap_{i \in I} F_i \neq \emptyset \). Now if \( X \) is not compact, then there is an open cover \( \{U_i\}_{i \in I} \) of \( X \) such that for any finite subset \( J \subset I \) we have \( X \neq \bigcup_{i \in J} U_i \). Define \( F_i = X \setminus U_i \). Then \( F_i \) is closed and \( \bigcap_{i \in I} F_i = X \setminus \bigcup_{i \in I} U_i = \emptyset \). Hence \( \{F_i\} \) does not have finite intersection property. That is there is a finite subset \( J \subset I \) so that \( \bigcap_{i \in J} F_i = \emptyset \). But then \( \bigcup_{i \in J} U_i = X \setminus \bigcap_{i \in I} F_i = X \), contradiction.

4. We can construct a sequence of continuous functions \( f_n \) in \( A \), where \( f_n(x) = 0 \) for \( 0 \leq x \leq \frac{1}{n+1} \), \( f_n(x) = 1 \) for \( \frac{1}{n} \leq x \leq 1 \), and \( f_n(x) = n(n+1)(x - \frac{1}{n+1}) \) is linear for \( \frac{1}{n+1} \leq x \leq \frac{1}{n} \). Then for \( m > n \), \( f_m(\frac{1}{m}) = 1, f_n(\frac{1}{m}) = 0 \) so \( d(f_n, f_m) \geq 1 \). But any \( 1/2 \)-ball contains at most one \( f_n \), so \( A \) has no finite \( 1/2 \)-net.

5. Let \( \varepsilon > 0 \). There exists a finite \( \varepsilon/2 \)-net for \( A \), \( S = \{x_1, \ldots, x_n\} \). Then \( A \subset \bigcup_{i \leq n} B(x_i, \varepsilon/2) \subset \bigcup_{i \leq n} \overline{B}(x_i, \varepsilon/2) \). Since \( \bigcup_{i \leq n} \overline{B}(x_i, \varepsilon/2) \) is closed (as a finite union of closed sets), it follows that \( A \subset \bigcup_{i \leq n} \overline{B}(x_i, \varepsilon/2) \subset \bigcup_{i \geq n} B(x_i, \varepsilon) \). So, \( S = \{x_1, \ldots, x_n\} \) is a finite \( \varepsilon \)-net for \( A \).

6. Assume that \( (X, d) \) is totally bounded. Let \( \{x_n\} \) be any sequence in \( X \). Since \( (X, d) \) is totally bounded, \( X \) can be covered by finitely many balls of radius \( 1 \). One of these balls, say \( B(y_1, 1) \), contains \( x_n \) for infinitely many \( n \)'s. Choose \( n_1 \in \mathbb{N} \) such that \( x_{n_1} \in B(y_1, 1) \). Since \( X \) can be covered by finitely many balls of radius \( 1/2 \), \( B(y_1, 1) \cap B(y_2, 1/2) \) for some \( y_2 \) contains \( x_{n_1} \) for infinitely many \( n \)'s. Choose \( n_2 > n_1 \) such that \( x_{n_2} \in B(y_1, 1) \cap B(y_2, 1/2) \). Proceeding in this way we find a sequence \( \{n_k\} \) of positive integers such that \( n_{k+1} > n_k \) for all \( k \), a sequence of balls \( B(y_k, 1/k) \), and a subsequence \( \{x_{n_k}\} \) satisfying \( x_{n_k} \in B(y_1, 1) \cap \cdots \cap B(y_k, 1/k) \). We claim that \( \{x_{n_k}\} \) is Cauchy. To see this note that for \( l \geq k \), we have \( x_{n_l}, x_{n_l} \in B(y_k, 1/k) \), hence \( d(x_{n_l}, x_{n_l}) < 2/k \). Take \( \varepsilon > 0 \) and choose \( N \in \mathbb{N} \) such that \( 2/N \leq \varepsilon \). Then for \( l, k \geq N \), \( d(x_{n_l}, x_{n_l}) < 2/N \leq \varepsilon \). Consequently, \( \{x_n\} \) has a Cauchy subsequence.
Conversely, arguing by contradiction, assume that $(X,d)$ is not totally bounded. Then there is $\alpha$ such that $X$ is not equal to a finite union of balls of radius $\alpha$. Take any point $x \in X$ and call it $x_1$. Then there exists a point $x_2 \notin B(x_1,\alpha)$. So $d(x_2,x_1) \geq \alpha$. Since $X \neq B(x_1,\alpha) \cup B(x_2,\alpha)$, there exists $x_3 \notin B(x_1,\alpha) \cup B(x_2,\alpha)$. That is $d(x_3,x_1) \geq \alpha$ and $d(x_3,x_2) \geq \alpha$. Continuing this way we find a sequence $\{x_n\}$ such that $x_{n+1} \notin B(x_1,\alpha) \cup \cdots \cup B(x_n,\alpha)$. That is, $d(x_{n+1},x_1) \geq \alpha, \ldots, d(x_{n+1},x_n) \geq \alpha$. The sequence $\{x_n\}$ has the property that $d(x_m,x_n) \geq \alpha$ for all $n \neq m$, so it does not contain a Cauchy subsequence.

7. Let $\varepsilon > 0$. Since $f$ is uniformly continuous there is $\delta > 0$ such that if $d(x,y) < \delta$, then $d(f(x),f(y)) < \varepsilon$. The space $X$ is totally bounded so there is a finite set of points $x_1, \ldots, x_k$ such that $X = \bigcup_{i=1}^{k} B(x_i,\delta)$. We claim that $Y = \bigcup_{i=1}^{k} B(f(x_i),\varepsilon)$. To see this, let $y \in Y$, then there is $x \in X$ such that $f(x) = y$. Since $X = \bigcup_{i=1}^{k} B(x_i,\delta)$, $x \in B(x_i,\delta)$ for some $1 \leq i \leq k$. So $y = f(x) \in f(B(x_i,\delta)) \subset B(f(x_i),\varepsilon)$. That is, $Y = \bigcup_{i=1}^{k} B(f(x_i),\varepsilon)$. (As you have noticed, in order to prove the result one only needs that $f : X \to Y$ is surjective and uniformly continuous. So the problem should be stated as follows: If $f : X \to Y$ is a bijection and $f, f^{-1}$ are uniformly continuous, then $X$ is totally bounded if and only if $Y$ is totally bounded.)

8. Let $\{U_i\}_{i \in I}$ be an open cover of the compact metric space $X$. We first show: there exists $\delta > 0$ such that every ball of radius $\delta$ is contained in $U_i$ for some $i \in I$.

Proof. Assume not. Then we can construct a sequence of balls $B(x_n,1/n)$ each of which is not contained in any $U_i$. Since $X$ is compact, the sequence $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to x$. But $x \in U_i$ for some $i \in I$ and $U_i$ is open. So there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U_i$. Now consider $y \in B(x_{n_k},1/n_k)$. For $k$ large, $d(x,y) \leq d(x,x_{n_k}) + d(x_{n_k},y) < d(x,x_{n_k}) + 1/n_k < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $B(x_{n_k},1/n_k) \subset B(x,\varepsilon) \subset U_i$ for $k$ large, contradiction. Finally, let $A \subset X$ with diam $(A) < \delta$ and pick $x \in A$. Then $A \subset B(x,\delta)$, hence $A \subset U_i$ for some $i \in I$.

9. Clearly $f$ is injective. To see that it is also surjective we argue by a contradiction and assume that there is $a \in X \setminus f(X)$. Since $f(X)$ is compact, $d(a,f(X)) = r > 0$. Define the sequence $x_1 = f(a), x_2 = f^2(a), x_3 = f^3(a), \ldots$. For $m > n$ we have $d(x_m,x_n) = d(f^m(a),f^n(a)) = d(f^{m-1}(a),f^{n-1}(a)) = \cdots = d(f^{m-n}(a),a) \geq r > 0$ since $f^{m-n}(a) \in f(X)$ and $d(a,f(X)) = r$. Hence, no subsequence of $\{x_n\}$ is Cauchy, so no subsequence converges, contradicting the fact that $X$ is compact.