Some solutions to Problem Set 7.

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1. None of the spaces is compact. Take the sequence $\{x_n\}$ with $x_n = n$. Then $\{x_n\}$ does not have a convergent subsequence in $\mathbb{R}$ with the metrics d_1 and d_2. (Alternatively: $(\mathbb{R}, d_1)$ is equivalent to $\mathbb{R}$ with the usual metric and $(\mathbb{R}, d_2)$ is isometric to $(-\pi/2, \pi/2)$ with the usual metric.) Also $d_3(x_n, x_m) = 1$ for $n \neq m$, so the sequence does not have a convergent subsequence with the metric d_3.

2. Let $\varepsilon > 0$. For each $x \in X$, there exists $\delta_x > 0$ such that $d(x, y) < 2\delta_x$ implies $d(f(x), f(y)) < \frac{1}{2}\varepsilon$. Now $\{B(x, \delta_x) : x \in X\}$ is an open cover of X so contains a finite subcover, say $\{B(x_1, \delta_{x_1}), \ldots, B(x_n, \delta_{x_n})\}$. Let $\delta = \min(\delta_{x_1}, \ldots, \delta_{x_n})$. Now let $y, z \in X$ with $d(y, z) < \delta$. Then $y \in B(x_i, \delta_{x_i})$ for some i in $\{1, \ldots, n\}$ so $d(y, x_i) < \delta_{x_i}$ and $d(z, x_i) \leq d(z, y) + d(y, x_i) < \delta + \delta_{x_i} \leq \delta_{x_i} + \delta_{x_i} = 2\delta_{x_i}$. Thus $d(f(z), f(y)) \leq d(f(z), f(x_i)) + d(f(x_i), f(y)) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. This proves that f is uniformly continuous.

3. Arguing by contradiction assume that $\{F_i\}_{i \in I}$ is a family of closed sets having the finite intersection property such that $\bigcap_{i \in I} F_i = \emptyset$. Let $U_i = X \setminus F_i$. Then the U_i's are open and $\bigcup_{i \in I} U_i = \bigcup_{i \in I}(X \setminus F_i) = X \setminus \bigcap_{i \in I} F_i = X$. Since X is compact, there is a finite set $J \subset I$ such that $X = \bigcup_{i \in J} U_i$. But then $\bigcap_{i \in J} F_i = \bigcap_{i \in J}(X \setminus U_i) = X \setminus \bigcup_{i \in J} U_i = \emptyset$, contradicting the fact that $\bigcap_{i \in J} F_i \neq \emptyset$ for any finite J. Conversely, assume that for every family of closed sets $\{F_i\}_{i \in I}$ having finite intersection property, we have $\bigcap_{i \in I} F_i \neq \emptyset$. Now if X is not compact, then there is an open cover $\{U_i\}_{i \in I}$ of X such that for any finite subset $J \subset I$ we have $X \neq \bigcup_{i \in J} U_i$. Define $F_i = X \setminus U_i$. Then F_i is closed and $\bigcap_{i \in I} F_i = X \setminus \bigcup_{i \in I} U_i = \emptyset$. Hence $\{F_i\}$ does not have finite intersection property. That is there is a finite subset $J \subset I$ so that $\bigcap_{i \in J} F_i = \emptyset$. But then $\bigcup_{i \in J} U_i = X \setminus \bigcap_{i \in J} F_i = X$, contradiction.

4. We can construct a sequence of continuous functions f_n in A, where $f_n(x) = 0$ for $0 \leq x \leq \frac{1}{n+1}$, $f_n(x) = 1$ for $\frac{1}{n} \leq x \leq 1$, and $f_n(x) = n(n+1)(x - \frac{1}{n+1})$ is linear for $\frac{1}{n+1} \leq x \leq \frac{1}{n}$. Then for $m > n$, $f_m(\frac{1}{m}) = 1, f_n(\frac{1}{m}) = 0$ so $d(f_n, f_m) \geq 1$. But any $1/2$-ball contains at most one f_n, so A has no *finite* $1/2$-net.

5. Let $\varepsilon > 0$. There exists a finite $\varepsilon/2$-net for A, $S = \{x_1, \ldots, x_n\}$. Then $A \subset \bigcup_{i=1}^n B(x_i, \varepsilon/2) \subset \bigcup_{i=1}^n \overline{B}(x_i, \varepsilon/2)$. Since $\bigcup_{i=1}^n \overline{B}(x_i, \varepsilon/2)$ is closed (as a finite union of closed sets), it follows that $\overline{A} \subset \bigcup_{i=1}^n \overline{B}(x_i, \varepsilon/2) \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$. So, $S = \{x_1, \ldots, x_n\}$ is a finite ε-net for $\overline{A}$.

6. Assume that (X, d) is totally bounded. Let $\{x_n\}$ be any sequence in X. Since (X, d) is totally bounded, X can be covered by finitely many balls of radius 1. One of these balls, say $B(y_1, 1)$, contains x_n for infinitely many n's. Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in B(y_1, 1)$. Since X can be covered by finitely many balls of radius $1/2$, $B(y_1, 1) \cap B(y_2, 1/2)$ for some y_2 contains x_n for infinitely many n's. Choose $n_2 > n_1$ such that $x_{n_2} \in B(y_1, 1) \cap B(y_2, 1/2)$. Proceeding in this way we find a sequence $\{n_k\}$ of positive integers such that $n_{k+1} > n_k$ for all k, a sequence of balls $B(y_k, 1/k)$, and a subsequence $\{x_{n_k}\}$ satisfying $x_{n_k} \in B(y_1, 1) \cap \cdots \cap B(y_k, 1/k)$. We claim that $\{x_{n_k}\}$ is Cauchy. To see this note that for $l \geq k$, we have $x_{n_k}, x_{n_l} \subset B(y_k, 1/k)$, hence $d(x_{n_k}, x_{n_l}) < 2/k$. Take $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $2/N \leq \varepsilon$. Then for $l, k \geq N$, $d(x_{n_k}, x_{n_l}) < 2/N \leq \varepsilon$. Consequently, $\{x_n\}$ has a Cauchy subsequence.

Conversely, arguing by contradiction, assume that (X, d) is not totally bounded. Then there is α such that X is not equal to a finite union of balls of radius α. Take any point $x \in X$ and call it x_1. Then there exists a point $x_2 \notin B(x_1, \alpha)$. So $d(x_2, x_1) \geq \alpha$. Since $X \neq B(x_1, \alpha) \cup B(x_2, \alpha)$, there exists $x_3 \notin B(x_1, \alpha) \cup B(x_2, \alpha)$. That is $d(x_3, x_1) \geq \alpha$ and $d(x_3, x_2) \geq \alpha$. Continuing this way we find a sequence $\{x_n\}$ such that $x_{n+1} \notin B(x_1, \alpha) \cup \cdots \cup B(x_n, \alpha)$. That is, $d(x_{n+1}, x_1) \geq \alpha, \ldots, d(x_{n+1}, x_n) \geq \alpha$. The sequence $\{x_n\}$ has the property that $d(x_m, x_n) \geq \alpha$ for all $n \neq m$, so it does not contain a Cauchy subsequence.

7. Let $\varepsilon > 0$. Since f is uniformly continuous there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$. The space X is totally bounded so there is a finite set of points $x_1, \cdots, x_k$ such that $X = \bigcup_{i=1}^k B(x_i, \delta)$. We claim that $Y = \bigcup_{i=1}^k B(f(x_i), \varepsilon)$. To see this, let $y \in Y$, then there is $x \in X$ such that $f(x) = y$. Since $X = \bigcup_{i=1}^k B(x_i, \delta)$, $x \in B(x_i, \delta)$ for some $1 \leqslant i \leqslant k$. So $y = f(x) \in f(B(x_i, \delta)) \subset B(f(x_i), \varepsilon)$. That is, $Y = \bigcup_{i=1}^k B(f(x_i), \varepsilon)$. (As you have noticed, in order to prove the result one only needs that $f : X \to Y$ is surjective and uniformly continuous. So the problem should be stated as follows: If $f : X \to Y$ is a bijection and f, f^{-1} are uniformly continuous, then X is totally bounded if and only if Y is totally bounded.)

8. Let $\{U_i\}_{i \in I}$ be an open cover of the compact metric space X. We first show: there exists $\delta > 0$ such that every ball of radius δ is contained in U_i for some $i \in I$.
Proof. Assume not. Then we can construct a sequence of balls $B(x_n, 1/n)$ each of which is not contained in any U_i. Since X is compact, the sequence $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to x$. But $x \in U_i$ for some $i \in I$ and U_i is open. So there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U_i$. Now consider $y \in B(x_{n_k}, 1/n_k)$. For k large, $d(x, y) \leq d(x, x_{n_k}) + d(x_{n_k}, y) < d(x, x_{n_k}) + 1/n_k < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $B(x_{n_k}, 1/n_k) \subset B(x, \epsilon) \subset U_i$ for k large, contradiction.
Finally, let $A \subset X$ with diam $(A) < \delta$ and pick $x \in A$. Then $A \subset B(x, \delta)$, hence $A \subset U_i$ for some $i \in I$.

9. Clearly f is injective. To see that it is also surjective we argue by a contradiction and assume that there is $a \in X \setminus f(X)$. Since $f(X)$ is compact, $d(a, f(X)) = r > 0$. Define the sequence $x_1 = f(a), x_2 = f^2(a), x_3 = f^3(a), \cdots$. For $m > n$ we have

$$d(x_m, x_n) = d(f^m(a), f^n(a)) = d(f^{m-1}(a), f^{n-1}(a)) = \cdots = d(f^{m-n}(a), a) \geq r > 0$$

since $f^{m-n}(a) \in f(X)$ and $d(a, f(X)) = r$. Hence, no subsequence of $\{x_n\}$ is Cauchy, so no subsequence converges, contradicting the fact that X is compact.