

Some solutions to Problem Set 8.

1. (a) $\mathcal{T}_1 = \{\emptyset, X\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$, $\mathcal{T}_3 = \{\emptyset, \{b\}, X\}$, $\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, X\}$.
 (b) Closures of $\{a\}$ are: X , X , $\{a\}$, $\{a\}$ and interiors of $\{a\}$ are: \emptyset , $\{a\}$, \emptyset , $\{a\}$.
 Closures of $\{b\}$ are: X , $\{b\}$, X , $\{b\}$ and interiors of $\{b\}$ are: \emptyset , \emptyset , $\{b\}$, $\{b\}$.
 (c) (X, \mathcal{T}_2) , (X, \mathcal{T}_3) are homeomorphic (and each space is homeomorphic to itself.)
 (d) Only (X, \mathcal{T}_4) is Hausdorff (as all points must be closed in a Hausdorff space).
2. (a) Clearly, \emptyset and $X \in \mathcal{T}$. Let $\{A_i\}_{i \in I} \subseteq \mathcal{T}$. If $A_i = \emptyset$ for all i , then $\bigcup_{i \in I} A_i = \emptyset \in \mathcal{T}$. Otherwise we have $X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} [X \setminus A_i] \subset X \setminus A_k$ for any $k \in I$. The set $X \setminus \bigcup_{i \in I} A_i$ is countable as a subset of a countable set $X \setminus A_k$, hence $\bigcup_{i \in I} A_i \in \mathcal{T}$. Let $A_1, \dots, A_n \in \mathcal{T}$. If some $A_i = \emptyset$, then $\bigcap_{1 \leq i \leq n} A_i = \emptyset \in \mathcal{T}$. Otherwise, consider $X \setminus \bigcap_{1 \leq i \leq n} A_i = \bigcup_{1 \leq i \leq n} [X \setminus A_i]$. Since a finite union of countable sets is countable, $X \setminus \bigcap_{1 \leq i \leq n} A_i$ is countable and so $\bigcap_{1 \leq i \leq n} A_i \in \mathcal{T}$.
 (b) Clearly, \emptyset and $\mathbb{R} \in \mathcal{T}$. Let $\{A_i\}_{i \in I} \subseteq \mathcal{T}$. If all $A_i = \emptyset$ then $\bigcup_i A_i = \emptyset$. If one of the A_i 's is equal to \mathbb{R} , then $\bigcup_i A_i = \mathbb{R}$. So we may assume that $A_i = (a_i, \infty)$ for all $i \in I$. If the set $\{a_i \mid i \in I\}$ is not bounded from below, then $\bigcup_i A_i = \mathbb{R}$. If it is bounded, then $\bigcup_i A_i = (a, \infty)$ where $a = \inf\{a_i \mid i \in I\}$. Similarly if $A_1, \dots, A_n \in \mathcal{T}$, then $\bigcap_{1 \leq i \leq n} A_i \in \mathcal{T}$.
3. If $x \in \mathbb{R}$, then $x \in [x, x+1) \in \mathcal{B}$. Let $x \in [a_1, b_1) \cap [a_2, b_2)$. Then $x \in [a, b)$, where $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$. So \mathcal{B} is a basis for a topology on \mathbb{R} . Next we will show that $\mathcal{T} \subset \mathcal{T}_l$ but $\mathcal{T} \neq \mathcal{T}_l$, where \mathcal{T} is the usual topology on \mathbb{R} . Consider (a, b) . Then there is N such that $a + 1/N < b$, and $(a, b) = \bigcup_{n \geq N} [a + 1/n, b) \in \mathcal{T}_l$. Hence $\mathcal{T} \subset \mathcal{T}_l$ since the open intervals (a, b) form a basis for the usual topology \mathcal{T} . Consider $[a, b) \in \mathcal{T}_l$. The point a is not an interior point of $[a, b)$ in $(\mathbb{R}, \mathcal{T})$ since $(c, d) \cap [a, b)^c \neq \emptyset$ for any $(c, d) \in \mathcal{T}$ containing a . So $[a, b) \notin \mathcal{T}$, and $\mathcal{T}_l \neq \mathcal{T}$.
 $[a, b)$ closed since $[a, b)^c = (-\infty, a) \cup [b, \infty)$ is open. Indeed, $(-\infty, a) = \bigcup_{n \geq 1} [-n, a)$ is open since it is a union of open sets, and similarly, $[b, \infty)$ is open since $[b, \infty) = \bigcup_{n \geq 1} [b, b+n)$. Hence $\overline{[a, b)} = [a, b)$.
 (a, b) is not closed, since $(a, b)^c = (-\infty, a] \cup [b, \infty)$ is not open. Indeed if $a \in U = \bigcup_i [a_i, b_i)$ then $a \in [a_i, b_i)$ for some i , hence there exists $x \in U$ such that $a < x < b$, which is impossible. So $\overline{(a, b)} = [a, b)$ is the smallest closed set containing (a, b) .
 $[a, b]$ is closed since $[a, b]^c = (-\infty, a) \cup (b, \infty)$ is open in \mathcal{T} , hence open in \mathcal{T}_l . Hence, $\overline{[a, b]} = [a, b]$.
 $(a, b]$ is not closed, since $(a, b]^c = (-\infty, a) \cup (b, \infty)$ is not closed by the same argument as above. Hence $\overline{(a, b]} = [a, b]$ is the smallest closed set containing $(a, b]$.
4. Clearly, \emptyset and $\mathbb{R} \in \mathcal{T}$. Let $\{A_i\}_{i \in I} \subseteq \mathcal{T}$. If one of the sets $A_i = \mathbb{R}$, then the union is equal to \mathbb{R} . If none of them is equal to \mathbb{R} , then $0 \notin A_i$ for all i so that $0 \notin \bigcup_{i \in I} A_i$. In either case, $\bigcup_{i \in I} A_i \in \mathcal{T}$. If $A_1, \dots, A_n \in \mathcal{T}$, and $0 \notin A_i$ for some k , then $0 \notin \bigcap_{i \in I} A_i$. If $0 \in A_i$ for all i , then $A_i = \mathbb{R}$ for all i , and so $\bigcap_{i \in I} A_i = \mathbb{R}$. Hence \mathcal{T} is a topology on \mathbb{R} . Closed sets in $(\mathbb{R}, \mathcal{T})$ are complements of open sets. So \emptyset and all sets $C \subseteq \mathbb{R}$ such that $0 \in C$ are closed. The closure of $\{1\}$ is the smallest closed set containing $\{1\}$. Since any non-empty closed set has to contain 0 , $\overline{\{1\}} = \{0, 1\}$.
 The topology \mathcal{T} is not Hausdorff since if $x = 0$, then the only open set containing x is \mathbb{R} , and for any open set A containing $y \neq 0$, $\mathbb{R} \cap A = A \neq \emptyset$.

5. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$. So $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, we have $A \cup B \subseteq \overline{A} \cup \overline{B}$. The set on the right side is closed, so $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. Conversely, $A \subseteq A \cup B$ and $B \subseteq A \cup B$. So $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A \cup B}$. Consequently, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

6. Since $A^\circ \subseteq A$, $X \setminus A \subseteq X \setminus A^\circ$. Hence, $\overline{X \setminus A} \subseteq \overline{X \setminus A^\circ} = X \setminus A^\circ$ since $X \setminus A^\circ$ is closed (A° is open). Conversely, let B be any closed set containing $X \setminus A$. Then $X \setminus B$ is open and $X \setminus B \subseteq A$. Hence $X \setminus B \subseteq A^\circ$ since A° is the largest open set contained in A . Then, taking complements, $X \setminus A^\circ \subseteq B$. Since this holds for any closed set containing $X \setminus A$, we have $X \setminus A^\circ \subseteq \overline{X \setminus A}$. The proof of $(X \setminus A)^\circ = X \setminus \overline{A}$ is similar.

7. Let $f : X \rightarrow \mathbb{R}$ be continuous. If there are $a, b \in f(X)$ so that $a \neq b$, take open disjoint intervals I and J such that $a \in I$, $b \in J$. Then the sets $f^{-1}(I)$ and $f^{-1}(J)$ are open and disjoint. So $\mathbb{R} \setminus f^{-1}(I)$ and $\mathbb{R} \setminus f^{-1}(J)$ are finite, hence $[\mathbb{R} \setminus f^{-1}(I)] \cup [\mathbb{R} \setminus f^{-1}(J)]$ is finite. But $[\mathbb{R} \setminus f^{-1}(I)] \cup [\mathbb{R} \setminus f^{-1}(J)] = \mathbb{R} \setminus [f^{-1}(I) \cap f^{-1}(J)] = \mathbb{R}$, contradiction.

8. Assume that $f : X \rightarrow Y$ is continuous. Let $U \in \mathcal{B}$. Then U is open in Y and so, $f^{-1}(U)$ is open in X . Conversely, suppose that $f^{-1}(U)$ is open in X for every $U \in \mathcal{B}$. Take an open set $W \subseteq Y$. Then $W = \bigcup_{x \in W} U_x$, where $U_x \in \mathcal{B}$. Then $f^{-1}(W) = f^{-1}(\bigcup_{x \in W} U_x) = \bigcup_{x \in W} f^{-1}(U_x)$. By assumption, each $f^{-1}(U_x)$ is open in X , so $\bigcup_{x \in W} f^{-1}(U_x)$ is open in X , and $f^{-1}(W)$ is open in X . Hence f is continuous.

9. Assume $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are homeomorphisms. Thus f and g are continuous functions with continuous inverses $f^{-1} : Y \rightarrow X$ and $g^{-1} : Z \rightarrow Y$. Then $g \circ f : X \rightarrow Z$ is a composition of continuous functions, so is continuous. Further $f^{-1} \circ g^{-1}$ is the inverse of $(g \circ f)$ since $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \text{identity} : X \rightarrow X$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \text{identity} : Z \rightarrow Z$. Thus $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is a composition of continuous maps, so is continuous. Hence $g \circ f$ is a homeomorphism.

10. (a) Take $f(x) = \frac{d-c}{b-a}(x-a) + c$. Then $f : (a, b) \rightarrow (c, d)$ is a bijection. As f and its inverse $f^{-1}(x) = \frac{b-a}{d-c}(x-c) + a$ are continuous, f is a homeomorphism.

Note that g defined by $g(x) = -1/x - 1$ is a homeomorphism from $(-1, 0)$ onto $(0, \infty)$, with inverse $g^{-1}(x) = -1/(x+1)$. By the previous part there is a homeomorphism from (a, b) to $(-1, 0)$, for example take $f(x) = \frac{-1}{b-a}(x-a)$. Then $g \circ f : (a, b) \rightarrow (0, \infty)$ is a homeomorphism. Finally, take $h(x) = g \circ f(x) + c$. Then $h : (a, b) \rightarrow (c, \infty)$ is a homeomorphism.

The map $g(x) = \tan x$ is a homeomorphism from $(-\pi/2, \pi/2)$ onto \mathbb{R} . The map $f(x) = \frac{\pi}{b-a}(x-a) - \pi/2$ is a homeomorphism from (a, b) onto $(-\pi/2, \pi/2)$, and so $h(x) = g \circ f(x) = \tan \left[\frac{\pi}{b-a}(x-a) - \pi/2 \right]$ is a homeomorphism from (a, b) onto \mathbb{R} .

(b) For $a = (x, y) \neq 0$, define

$$f(a) = g(\|a\|)a$$

where $g(r) = (1+r)/r$ for $r > 0$. Then $\|f(a)\| = g(\|a\|)\|a\| = 1 + \|a\| > 1$ and so $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2 \setminus \overline{B}((0, 0), 1)$. If $f(a) = f(b)$, then since g is increasing, $\|a\| = \|b\|$ and so $a = b$. If $\|b\| > 1$, then let $a = [(\|b\| - 1) \cdot b / \|b\|]$. Then $f(a) = b$. So f is a surjection, and since it is also an injection, it is a bijection with inverse $f^{-1}(b) = [(\|b\| - 1) \cdot b / \|b\|]$. Both maps f, f^{-1} are continuous. So f is a homeomorphism.