Some solutions to Problem Set 9.

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1. Call a space X discrete if it is equipped with the discrete topology and trivial if it is equipped with the the trivial topology. Now prove:
(a) If X is discrete, then every function $f : X \to Y$, where Y is any topological spaces, is continuous.
(b) If X is trivial with at least two elements, then there exists a topological space Y and a function $f : X \to Y$ that is not continuous.
(c) If Y is trivial, then every function $f : X \to Y$, where X is any topological space, is continuous.
(d) If Y is discrete and contains at least two elements, then there exists a topological space X and a function $f : X \to Y$ that is not continuous.

Solution: (b) For example, take $Y = X$ with the discrete topology. Then the identity function $f : X = (X, \mathit{trivial}) \to Y = (X, \mathit{discrete})$ is not continuous.
(d) For example, take $X = Y$ with the trivial topology. Again the identity function $f : X = (Y, \mathit{trivial}) \to Y = (Y, \mathit{discrete})$ is not continuous.

2. Show that the unit open ball $B = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ in $\mathbb{R}^2$ is homeomorphic to the open square $C = (-1, 1) \times (-1, 1)$ using the usual topology.

Solution: Note that $f(x, y) = \frac{\sqrt{x^2+y^2}}{\max\{|x|,|y|\}} \cdot (x, y)$ takes the circle $\partial B(0, r)$ onto the boundary ∂C_r of the square $C_r = (-r, r) \times (-r, r)$. The set $B(0, 1) \setminus \{(0, 0)\}$ is the disjoint union of $\partial B(0, r)$, $0 < r < 1$, and $C \setminus \{(0, 0)\}$ is the disjoint union of ∂C_r, $0 < r < 1$. Hence define $F : B \to C$ by $F(0, 0) = (0, 0)$ and for $(x, y) \neq (0, 0)$, $F(x, y) = \frac{\sqrt{x^2+y^2}}{\max\{|x|,|y|\}} \cdot (x, y)$. Its inverse is $F^{-1} : C \to B$ is given by $F^{-1}(0, 0) = (0, 0)$ and otherwise $F^{-1}(x, y) = \frac{\max\{|x|,|y|\}}{\sqrt{x^2+y^2}} \cdot (x, y)$. Clearly, both F, F^{-1} are continuous at points where $(x, y) \neq (0, 0)$. To see that F and F^{-1} is continuous at $(0, 0)$, first note that there are positive constants α and β such that $\alpha \max\{|x|, |y|\} \leqslant \sqrt{x^2 + y^2} \leqslant \beta \max\{|x|, |y|\}$. (Recall that the metrics d_2 and d_∞ are Lipschitz equivalent). Using this we get $\big\|F(x, y)\big\| \leqslant \beta\big\|(x, y)\big\| \to 0$ and $\big\|F^{-1}(x, y)\big\| \leqslant \alpha\big\|(x, y)\big\| \to 0$ as $\big\|(x, y)\big\| \to 0$. Thus $F(x, y) \to (0, 0) = F(0, 0)$ and $F^{-1}(x, y) \to (0, 0) = F^{-1}(0, 0)$ as $(x, y) \to (0, 0)$, so F, F^{-1} are continuous at $(0, 0)$.

3. Show that the unit open ball $B = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ in $\mathbb{R}^2$ is homeomorphic to the open right half plane $H = (0, \infty) \times \mathbb{R}$ using the usual topology.

Solution: We first show that H is homeomorphic with $\mathbb{R}^2$. To see this take any strictly increasing homeomorphism $f : (0, \infty) \to \mathbb{R}$ (for example $f(x) = \ln x$). Define $F(x, y) = (f(x), y)$. Then $F : H \to \mathbb{R}^2$ is a bijection, and both F, F^{-1} are continuous. Next we show that $\mathbb{R}^2$ and the open ball B are homeomorphic. To do this, define $G : B \to \mathbb{R}^2$ by $G(x, y) = \frac{1}{1-\|(x,y)\|}(x, y)$. (Note that $\big\|G(x, y)\big\| \to \infty$ as $\big\|(x, y)\big\| \to 1$.) Then G is clearly continuous and has a continuous inverse $G^{-1} : \mathbb{R}^2 \to B$ given by $G^{-1}(x, y) = \frac{1}{1+\|(x,y)\|}(x, y)$. (Check!) The composition of homeomorphisms is again a homeomorphism. So the map $G^{-1} \circ F : H \to B$ gives the required homeomorphism.

4. Let $f, g : X \to Y$ be continuous functions between topological spaces, and assume that Y is Hausdorff. Prove that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

Solution: Let $A = \{x \in X : f(x) = g(x)\}$. It suffices to show that the complement A^c is open in X. So take $x \in A^c$. Then $f(x) \neq g(x)$ and since Y is Hausdorff there are two disjoint open sets V_x and W_x such that $f(x) \in V_x$ and $g(x) \in W_x$. Set $U_x := f^{-1}(V_x) \cap g^{-1}(W_x)$. Then U_x is open and $x \in U_x$. Further, if $y \in U_x$, then $f(y) \in V_x$ and $g(y) \in W_x$ showing that $f(y) \neq g(y)$. Hence $U_x \subset A^c$ so that $A^c = \bigcup_{x \in A^c} U_x$ is open as required.

5. Let $(X, \mathcal{T})$ be a compact topological space and let A, B be closed subsets of $(X, \mathcal{T})$. Show that $A \cup B$ is compact.

Solution: $A \cup B$ is a closed subset of a compact space, hence compact.

6. Let $X = (0, 1)$ and let

$$\mathcal{T} = \{A \subseteq X | \, A = \emptyset \text{ or } A = (0, 1) \text{ or } A = (0, 1 - 1/n) \text{ for } n \geq 2\}.$$

Show that every open set A other than X is compact. Is X compact?

Solution: Let $A \in \mathcal{T}$ and $A \neq (0, 1)$. Hence $A = (0, 1 - 1/k)$ for some $k \geq 1$. Suppose that $\{U_i\}_{i \in I}$ is an open cover of A. Here we can take $I \subseteq \mathbb{N}$. Since $(0, 1-1/k) \subseteq \bigcup_{i \in I} U_i$, there is $i \in I$ such that $(0, 1-1/k) \subseteq U_i$. Hence A is compact in this topology. Consider now the whole space $X = (0, 1)$. Then $(0, 1) = \bigcup_{n \geq 2} (0, 1-1/n)$. However, $(0, 1)$ cannot be covered by finitely many of such sets since for any $N \in \mathbb{N}$, $\bigcup_{2 \leqslant n \leqslant N} (0, 1 - 1/n) = (0, 1 - 1/N)$. So $(X, \mathcal{T})$ is not compact.

7. Let $\mathcal{T}$ be the co-countable topology on $\mathbb{R}$, that is,

$$\mathcal{T} = \{A \subseteq \mathbb{R} | \, A = \emptyset \text{ or } \mathbb{R} \setminus A \text{ is countable}\}.$$

Is $[0, 1]$ compact in $(\mathbb{R}, \mathcal{T})$? What are the compact sets in $(\mathbb{R}, \mathcal{T})$?

Solution: Clearly, if $F \subset \mathbb{R}$ is finite, then it is compact. If F is an infinite subset of $\mathbb{R}$, then it is not compact. Indeed, since F has infinitely many elements it contains a countable set of distinct points a_n, $n \in \mathbb{N}$. Define $U_n = \mathbb{R} \setminus \{a_k \mid k \geq n\}$. Then $\mathbb{R} \setminus U_n = \{a_k \mid k \geq n\}$ is countable so that U_n is open. Since $\bigcup_{n \geq 1} U_n = \mathbb{R}$, the collection is an open cover of F. If F were compact, then $F \subseteq U_1 \cup \cdots \cup U_N$ for some $N \in \mathbb{N}$. But $U_1 \cup \cdots \cup U_N = \mathbb{R} \setminus \{a_k \mid k \geq N\} = U_N$. So, $a_N \in F \subseteq U_1 \cup \cdots \cup U_N$ and $a_N \notin U_1 \cup \cdots \cup U_N$, contradiction.

8. Suppose that $\{X_i\}_{i \in I}$ is a family of non-empty closed sets X, and that at least one of the X_i is compact. In addition, assume that the family has the property that for any two $i, j \in I$ either $X_i \subset X_j$ or $X_j \subset X_i$. Show that $\bigcap_{i \in I} X_i \neq \emptyset$.

Solution: Say X_i is compact. Let $B = \{k \in I \,|\, X_i \subseteq X_k\}$. The $\bigcap_{k \in B} X_k = X_i$. Let $A = \{k \in I \,|\, X_i \supsetneq X_k\}$. Then each X_k is a closed subset of a compact space X_i and further $\bigcap_{k \in S} X_k \neq \emptyset$ for any finite subset S of A. This means that $\{X_k\}_{k \in A}$ has the finite intersection property. Hence $\bigcap_{k \in A} X_k \neq \emptyset$ since X_i is compact. So $\bigcap_{k \in I} X_k = (\bigcap_{k \in B} X_k) \cap (\bigcap_{k \in A} X_k) = (\bigcap_{k \in A} X_k) \neq \emptyset$.

9. Ask if you've tried this. (Barnsley, "Fractals Everywhere" has the details.)