620-311 Metric spaces, due Wednesday April 18 at 10 AM

Assignment 1

1. Let $A$ be an open subset of a metric space $(X, d)$.
   (a) Show, directly from the definition, that if $b \in A$ then $A \setminus \{b\}$ is open in $X$.
   (b) If $B$ is a finite subset of $A$ show, using (a) or otherwise, that $A \setminus B$ is open in $X$.
   (c) Deduce that every finite subset of $X$ is closed in $X$.

2. Let $X = C[0, 1]$ be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Recall that the supremum metric on $X$ is defined by
   \[ d_\infty(f, g) = \sup\{|f(x) - g(x)| : 0 \leq x \leq 1\} \]
   and the $L^1$ metric on $X$ is defined by
   \[ d_1(f, g) = \int_0^1 |f(x) - g(x)| \, dx. \]
   Consider the sequence $\{f_1, f_2, f_3, \ldots\}$ in $X$ where $f_n(x) = nx^n(1 - x)$ for $0 \leq x \leq 1$.
   (a) Determine whether $\{f_n\}$ converges in $(X, d_1)$.
   (b) Determine whether $\{f_n\}$ converges in $(X, d_\infty)$.
   (You may use any standard results about limits of real sequences.)

3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $(X \times Y, d)$ be the product metric space. Show that if $A \subseteq X$ and $B \subseteq Y$, then $A \times B = A \times B$.

4. Let $(X, d)$ be a metric space and let $A$ be a non-empty subset of $X$. Recall that for each $x \in X$, the distance from $x$ to $A$ is
   \[ d(x, A) = \inf\{d(x, a) : a \in A\}. \]
   (a) Prove that $\overline{A} = \{x \in X : d(x, A) = 0\}$.
   (b) Prove that $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$. [Hint: first show that $d(x, A) \leq d(x, y) + d(y, A)$.] 
   (c) Deduce the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.
   (d) Show that if $x \notin \overline{A}$ then $U = \{y \in X : d(y, A) < d(x, A)\}$ is an open set in $X$ such that $\overline{A} \subset U$ and $x \notin U$.

5. Determine whether the following sequences of functions converge uniformly.
   (a) $f_n(x) = e^{-nx^2}$, $x \in [0, 1]$;
   (b) $g_n(x) = e^{-x^2/n}$, $x \in [0, 1]$.

6. Let $X$ be the set of all real sequences with finitely many non-zero terms with the supremum metric: if $x = (x_i)$ and $y = (y_i)$ then $d(x, y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $x^n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$.
   (a) Show that $\{x^n\}$ is a Cauchy sequence in $X$.
   (b) Show that $\{x^n\}$ does not converge to a point in $X$. (So $X$ is not complete.)
7 Let $X$ be a nonempty set and let $(Y, d)$ be a complete metric space. Let $f : X \to Y$ be an injective function and define
\[ d_f(x, y) = d(f(x), f(y)) \]
for $x, y \in X$.

(a) Explain briefly why $d_f$ is a metric on $X$.
(b) Show that $(X, d_f)$ is a complete metric space if $f(X)$ is a closed subset of $Y$.

8 Let
\[ f(x) = \frac{2}{2+x} \quad \text{for} \ x \geq 0. \]

(a) Show that $f$ defines a contraction mapping $f : X \to X$ when $X = [0, \infty)$ is given the usual metric.
(b) Fix $x_0 \geq 0$ and $x_{n+1} = f(x_n)$ for all $n \geq 0$. Show that the sequence \( \{x_n\} \) converges and find its limit with respect to the usual metric on $\mathbb{R}$.

9 Let $X$ be a complete normed vector space over $\mathbb{R}$. (Recall a norm $\| \cdot \|$ satisfies $\|a + b\| \leq \|a\| + \|b\|$, $\|a\| \geq 0$ with $\|a\| = 0$ if and only if $a = 0$ and $\|\lambda a\| = |\lambda|\|a\|$. We then define a metric by $d(a, b) = \|a - b\|$.) A sphere in $X$ is a set
\[ S(a, r) = \{x \in X : d(x, a) = \|x - a\| = r\} \]
where $a \in X$ and $r > 0$.

(a) Show that each sphere in $X$ is nowhere dense.
(b) Show that there is no sequence of spheres $\{S_n\}$ in $X$ whose union is $X$.
(c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.

10. Let $(X, d)$ and $Y, d'$ be metric spaces and let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x \in X : f(x) = g(x)\}$ is a closed subset of $X$.
(b) Show that $f, g : X \to \mathbb{R}$ are continuous then $f - g$ is continuous and $\{x \in X : f(x) < g(x)\}$ is open.