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University of Melbourne

Assessment: Semester 1, 2006

Department of Mathematics and Statistics

620-321 Algebra

Exam duration — 3 hours

Reading time — 15 minutes

This paper consists of this cover page and two pages of examination questions.

Examination papers with common content: None.

Authorized materials: Pens, pencils, rubbers, rulers. No other materials are authorized. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator

Instructions to invigilators: Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the exam paper at the conclusion of the examination.

Instructions to students: This examination consists of 12 questions. All questions may be attempted. The number of marks for each question is approximately the same (even though the amount of work or sophistication may vary substantially). Answers should be appropriately justified. All of your calculations and working should be shown.

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BEGINNING OF EXAM QUESTIONS

1. (a) Give the definitions of Unique Factorization Domain (UFD) and Principal Ideal Domain (PID).

   (b) Give an example, if such exists, of

      i. a UFD that is not a PID;

      ii. a PID that is not a UFD.

2. (a) What are the units and zero-divisors in the ring $\mathbb{Z}_6$?

   (b) List all ideals in $\mathbb{Z}_6$.

3. Let R be a PID, and $p \in R$ a non-zero, non-unit element. Show that p is irreducible if and only if (p) is a maximal ideal in R.

4. Use the Euclidean algorithm to find the greatest common divisor in $\mathbb{Q}[x]$ of the polynomials $x^3 - 5x^2 + x - 5$ and $x^4 + x^3 + 2x^2 + x + 1$.

5. Factorize the polynomial $x^5 + 2x^4 + 4x^3 + 4x^2 + 3x + 2$ into irreducible factors in $F_5[x]$.

6. For each of the following polynomials, determine (i) whether it is irreducible in $\mathbb{Q}[x]$ and (ii) whether it is irreducible in $\mathbb{Z}[x]$. You should justify your answers.
   (a) $x^4 - 5x^2 - 30x - 15$ (b) $x^4 - x^2 - 2$ (c) $x^3 - 1$ (d) $2x^2 - 4$

7. (a) State the structure theorem for finitely generated modules over a PID.

   (b) Consider the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ for the free $\mathbb{Z}$-module $\mathbb{Z}^3$, and let $N \subset \mathbb{Z}^3$ be the submodule with basis $B_N = \{(2,2,2),(0,8,4)\}$. Find a new basis $\{f_1, f_2, f_3\}$ for $\mathbb{Z}^3$, and elements $d_1, d_2, d_3 \in \mathbb{Z}$ such that the non-zero elements of the set $\{d_1 f_1, d_2 f_2, d_3 f_3\}$ form a basis for N and $d_1 | d_2 | d_3$.
   [*Hint: Begin with the matrix whose columns are the elements of* B_N.]

8. (a) Find the invariant factor matrix over $\mathbb{Z}$ that is equivalent to the matrix:

$$\begin{bmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{bmatrix}$$

   (b) Give the primary decomposition and invariant factor decomposition of the $\mathbb{Z}$-module $\mathbb{Z}_{20} \oplus \mathbb{Z}_{40} \oplus \mathbb{Z}_{100}$.

9. Let V be an eight dimensional complex vector space, and $T : V \to V$ a linear transformation.

   (a) Explain how V can be regarded as a $\mathbb{C}[t]$-module.

   (b) Suppose that

$$V \cong \frac{\mathbb{C}[t]}{(t-2)(t-3)^2} \oplus \frac{\mathbb{C}[t]}{(t-2)(t-3)^3}$$

   as a $\mathbb{C}[t]$-module.

      i. What is the Jordan Normal Form of T?

      ii. What is the minimum polynomial of T?

      iii. What is the dimension of the eigenspace corresponding to the eigenvalue 3.

10. (a) Let $a \in \mathbb{R}$ be a constructible number. What can be said about the degree of the extension $\mathbb{Q}(a)$ of $\mathbb{Q}$?

   (b) Explain why it is not possible to construct with straight-edge and compass a line segment whose length is that of an edge of a cube of volume 5.

11. Let K be the splitting field of the polynomial $x^3 - 7 \in \mathbb{Q}[x]$.

   (a) Find the Galois group of K as an extension of $\mathbb{Q}$.

   (b) Exhibit an intermediate subfield L, $\mathbb{Q} \subseteq L \subseteq K$, such that L is *not* a Galois extension of $\mathbb{Q}$.

12. Let $K = \mathbb{Q}(\omega, \sqrt{5})$ where $\omega = e^{2\pi i/3}$.

   (a) Find a basis for K over $\mathbb{Q}$.

   (b) Show that K is a Galois extension of $\mathbb{Q}$, and describe the Galois group $G = G(K/\mathbb{Q})$.

   (c) Let $\beta = \omega + \sqrt{5} \in K$. Compute the orbit of β under the action of G on K.

END OF EXAMINATION