The University of Melbourne

Semester 1 Assessment 2007

COPY

Department of Mathematics and Statistics

620-321 Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.
This paper has: 3 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
Pens, pencils, rubbers, rulers. No other materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 9 questions. All questions may be attempted. Answers should be appropriately justified. The number of marks for each question is approximately the same (even though the amount of work or sophistication may vary substantially). All of your calculations and working should be shown.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Give the definitions of zero divisor, integral domain, and Euclidean domain.
   (b) Give an example of an integral domain that is not a Euclidean domain.
   (c) State the division algorithm as it applies to $\mathbb{Q}[x]$ and use it to prove that $\mathbb{Q}[x]$ is a Principal Ideal Domain.

2. Use the Euclidean algorithm to find a greatest common divisor d of $4 + 7i$ and $1 + 7i$ in $\mathbb{Z}[i]$. Find elements $x, y \in \mathbb{Z}[i]$ such that $d = x(4 + 7i) + y(1 + 7i)$.

3. (a) Factor $x^2 + 3$ into irreducible polynomials in $\mathbb{Z}_5[x]$ and $\mathbb{Z}_7[x]$.
   (b) Express $x^4 - x^2 - 2$ as a product of irreducibles in $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$ and $\mathbb{Z}_5[x]$.
   (c) Show that in a Unique Factorisation Domain all irreducible elements are prime.

4. List, up to isomorphism, all abelian groups of order 504. Give the primary decomposition and annihilator (as a $\mathbb{Z}$-module) of each group.

5. Let N be the submodule of (the $\mathbb{Z}$-module) $F = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by by

$$\{(1, 0, -1), (4, 3, -1), (0, 9, 3), (3, 12, 3)\}.$$

   (a) Find a basis $\{b_1, b_2, b_3\}$ of F and d_1, d_2, $d_3 \in \mathbb{Z}$ such that the the non-zero elements of the set $\{d_1b_1, d_2b_2, d_3b_3\}$ form a basis for N (as a $\mathbb{Z}$-module).
   (b) Give a direct sum of non-trivial cyclic groups that is isomorphic to F/N.

6. Let $A \in M_8(\mathbb{C})$ be a matrix and suppose that the matrix $xI - A \in M_8(\mathbb{C}[x])$ is equivalent to the matrix:

$$\operatorname{diag}(1, 1, 1, 1, (x - 1), (x - 1), (x - 1)(x - 2), (x - 1)(x - 2)^2(x - 3)).$$

   (a) Give the corresponding decomposition of $\mathbb{C}^8$ regarded as a $\mathbb{C}[x]$-module. (All summands should be non-trivial cyclic $\mathbb{C}[x]$-modules.)
   (b) Give the Jordan Normal Form of the matrix A.
   (c) Give the minimal and characteristic polynomials of A.

7. (a) Exactly on of the following is a field: $\mathbb{Z}_3[x]/(x^2-2)$ and $\mathbb{Z}_7[x]/(x^2+3)$. Decide which one. You should justify why it is a field and also why the other is not.
   (b) What is the order of the field F above?
   (c) Find a generator for the group F^* of non-zero elements of the field (under multiplication).

8. (a) i. Suppose that $a \in \mathbb{R}$ is a constructible number. What can be said about the dimension of $\mathbb{Q}(a)$ as a vector space over $\mathbb{Q}$?
      ii. Is it possible to construct a regular nonagon (9-gon) with straight-edge and compass? Explain. What about a regular hexagon (6-gon)?
   (b) Find the Galois groups of the following extensions K of $\mathbb{Q}$, and in each case list the correspondence between subgroups of $G(K/\mathbb{Q})$ and subfields $L \subseteq K$.
      i. $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$
      ii. $K = \mathbb{Q}(e^{2\pi i/3})$

   Explain why there is no proper subfield of $\mathbb{Q}(e^{2\pi i/3})$ other than $\mathbb{Q}$.

9. Let K be the splitting field of the polynomial $f(x) = x^4 - 3$
   (a) Calculate $[K : \mathbb{Q}]$.
   (b) Find a basis $\mathcal{B}$ for K as a $\mathbb{Q}$-vector space.
   (c) Give the size of the Galois group $G(K/\mathbb{Q})$, and identify it.
   (d) Write down a field L such that $\mathbb{Q} \subseteq L \subseteq K$ and L is not a Galois extension of $\mathbb{Q}$.

— END OF EXAMINATION QUESTIONS —