620-321 Algebra, Semester 1, 2009

Answers to Problem Sheet 1

1. Suppose that $R \neq \{0\}$. Let $x, y \in R^*$. Then $xy = x^2y \implies y = xy$ by the cancellation law. Similarly $x = xy$, and so $x = y$.

2. (a) $\{1, -1\}$
(b) $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$
(c) $\{1,2,3,4\}$
(d) $\{1,3\}$
(e) $\mathbb{Q} \setminus \{0\}$
(f) $\{1, -1, i, -i\}$
(g) $\{g \in \mathbb{R}[x] \mid g \text{ is non-zero}, \deg(g) = 0\}$

3. T, F, F, T, T, F, F

4. $\mathbb{Z}_2 \times \mathbb{Z}_2$

5. Suppose that F is a field. If I is an ideal and $0 \neq u \in I$, then u is a unit because F is a field. Hence $F = I$ (see exercise 14). Conversely, suppose R has exactly two ideals. One of these is of course $\{0\}$ and the other is R. So if $0 \neq a \in R$ then $(a) = aR = R$ and hence $1 \in aR$. Thus $1 = ab$ for some $b \in R$ and a is a unit. Since every non-zero element of R is a unit, R must be a field.

6. $\{0,3\} \subset \mathbb{Z}_6$

8. (a) $a0 = a(0+0) = a0 + a0$. Now subtract $a0$ from both sides. Similarly for $0a = 0$.
(b) $0 = 0b = (a + (-a))b = ab + (-a)b$. Hence $(-a)b = -(ab)$. Similarly for $a(-b) = -(ab)$.

10. (a) The map $r \to (r, 0)$ is an injective homomorphism. Note that it does not send 1_R to $1_{R\times S} = (1_R, 1_S)$.
(b) This map is an injective homomorphism.

11. (a) The rings are not isomorphic since $4(a,b) = (0,0)$ for any $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_4$ but $4 \cdot 1 = 4 \neq 0$ in $\mathbb{Z}_8$.
(b) We have two surjective homomorphisms $\varphi : \mathbb{Z} \to \mathbb{Z}_3$ and $\psi : \mathbb{Z} \to \mathbb{Z}_5$. Consider $\varphi \times \psi : \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_5$. The kernel of $\varphi \times \psi$ is $\ker \varphi \cap \ker \psi = (3) \cap (5) = (15)$. So this map induces an isomorphism $\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ (by the first isomorphism theorem).

12. That $I + J$ is again an ideal in R is easy. Consider $\varphi : R \to R/I \times R/J$ defined by $r \to (r + I, r + J)$. Clearly $\ker \varphi = I \cap J$. To show φ is onto consider an arbitrary element $(a + I, b + J) \in R/I \times R/J$. Suppose $I + J = R$. Then $1_R = i + j$ for some $i \in I, j \in J$. Put $c = bi + aj$. Now $c - a = c - a(i + j) = c - ai - aj = bi - ai \in I$. So $c \in a + I$. Similarly, $c - b = c - b(i + j) = c - bi - bj = aj - bj \in J$. So $c \in b + I$. Thus $\varphi(c) = (a + I, b + J)$ and so φ is onto. Hence $R/(I \cap J) \cong R/I \times R/J$.

    Here is an alternative proof that φ is onto. Since $I + J = R$ we can write $a = a_I + a_J$ where $a_I \in I$and $a_J \in J$. Similarly we can write $b = b_I + b_J$. Now $a + I = a_I + a_J + I = a_J + I$ and $b + J = b_I + b_J + J = b_I + J$. Then put $c = a_J + b_I$ and one can calculate that $\varphi(c) = (a + I, b + J)$.

13. If $\gcd(m, n) = 1$ then $(m) + (n) = \mathbb{Z}$. By the previous exercise with $I = (m), J = (n)$ we get $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ as required. Conversely suppose $\gcd(m, n) > 1$. Then $k = \operatorname{lcm}(m, n) < mn$ and $k(a, b) = 0$ for every $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Thus $\mathbb{Z}_{mn}$ has an element of order mn but $\mathbb{Z}_m \times \mathbb{Z}_n$ does not. So they cannot be isomorphic.

14. Since u is a unit, for some $v \in R$ we have $vu = uv = 1$. Thus $uv_1 = uv_2$ implies $v_1 = vuv_1 = vuv_2 = v_2$. If I is an ideal containing a unit u then for some $v \in R$ $vu = 1$ so $1 \in I$ and hence $I = R$. Conversely, if $I = R$ then I contains 1 which is a unit.