
620-321 Algebra, Semester 1, 2009
Answers to Problem Sheet 5

3. Suppose that N is generated by $\{n_1, \dots, n_p\}$ and that M/N is generated by $\{m_1 + N, \dots, m_q + N\}$. We claim that M is generated by $\{n_1, \dots, n_p, m_1, \dots, m_q\}$. For if $u \in M$ then $u + N = \sum_{i=1}^q a_i(m_i + N)$ for suitable $a_i \in R$ since the $m_i + N$ generate M/N . Hence $u - \sum_{i=1}^q a_i m_i \in N$ and since the n_i generate N , $u - \sum_{i=1}^q a_i m_i = \sum_{j=1}^p b_j n_j$ for suitable $b_j \in R$. Thus $u = -\sum_{i=1}^q a_i m_i + \sum_{j=1}^p b_j n_j$ and our claim is established.
4. Suppose that \mathbb{Q} is finitely generated as a \mathbb{Z} -module, say by the rational numbers $\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}$ where $a_i, b_i \in \mathbb{Z}$ and $b_i \neq 0$. Let $d = b_1 \cdots b_n$ and observe that each $\frac{a_i}{b_i}$ is an integer multiple of $\frac{1}{d}$ and so $\frac{1}{d}$ must generate \mathbb{Q} as well. Let p is a prime not dividing d . Then $\frac{1}{p}$ is not an integral multiple of $\frac{1}{d}$, since $\frac{1}{p} = \frac{m}{d}$ implies $d = mp$ while p does not divide d . So we have arrived at a contradiction. Hence \mathbb{Q} is not finitely generated.
5. (a) $M/N \cong \mathbb{Z} \oplus \mathbb{Z}_3$ and the torsion submodule is isomorphic to \mathbb{Z}_3 . The torsion submodule is generated by the coset $(0, 1) + N$.
- (b) $M/N \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$ so the whole module is torsion and isomorphic to \mathbb{Z}_6 . It is generated by the coset $(1, 1) + N$.
- (c) $M/N \cong \mathbb{Z}$ and the torsion submodule is trivial. M itself is generated by the elements $(2, 3), (1, 1)$ and hence these are a free basis. Thus the quotient is isomorphic to \mathbb{Z} .
6. (a) Yes. If M is a cyclic R -module, say with generator m so that $M = Rm$, then a quotient module $M/N = \{rm + N \mid r \in R\}$ by a submodule N will be generated as an R -module by the coset $m + N$.
- (b) No. Consider the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients viewed as a module over itself. Then $\mathbb{Z}[x]$ is a cyclic module, but the ideal generated by $\{2, x\}$ is not principal and so is not cyclic when viewed as a submodule.
7. (a) If $u = av_2 \in U$ then $x \cdot u = T(u) = T(av_2) = aT(v_2) = 2av_2 \in U$. Since U is a subspace closed under the action of x it is a $\mathbb{Q}[x]$ -submodule.
- (b) Observe that

$$x^2 \cdot v_1 = T(T(v_1)) = T(3v_1 - v_2) = 3T(v_1) - T(v_2) = 3(3v_1 - v_2) - 2v_2 = 9v_1 - 5v_2$$

and $x^2 \cdot v_2 = 4v_2$. Then one calculates that

$$\begin{aligned} f(x) \cdot v_1 &= (9v_1 - 5v_2) + 2(3v_1 - v_2) - 3v_1 = 12v_1 - 7v_2 \\ f(x) \cdot v_2 &= 4v_2 + 2(2v_2) - 3v_2 = 5v_2 \end{aligned}$$

11. (a) The map is surjective since, given a matrix (a_{ji}) , we can define a homomorphism α by setting $\alpha(f_i) = \sum_{j=1}^n a_{ji}f_j$ (see the alternative definition of a free module). It is also injective since

$$\begin{aligned} [\alpha] = 0 &\implies (a_{ji}) = 0 \quad \forall i, j \\ &\implies \alpha(f_i) = 0 \quad \forall i \\ &\implies \alpha(x) = 0 \quad \forall x \in F \quad \text{since the } f_i \text{ generate } F \\ &\implies \alpha = 0 \end{aligned}$$

We still have to show that the map is a (ring) homomorphism. First note that if we let $c = [\alpha \circ \beta]$, then $c_{ki} = \sum_{j=1}^n b_{ji}a_{kj}$ since

$$\alpha \circ \beta(f_i) = \alpha\left(\sum_{j=1}^n b_{ji}f_j\right) = \sum_{j=1}^n b_{ji}\alpha(f_j) = \sum_{j=1}^n b_{ji}\sum_{k=1}^n a_{kj}f_k = \sum_{k=1}^n \left(\sum_{j=1}^n b_{ji}a_{kj}\right) f_k.$$

On the other hand, the matrix product is given by $(ab)_{ki} = \sum_{j=1}^n a_{kj}b_{ji}$. Therefore $[\alpha \circ \beta]_{ki} = ([\alpha][\beta])_{ki}$ for all k and i , which is equivalent to $[\alpha \circ \beta] = [\alpha][\beta]$.

For addition we have $(\alpha + \beta)(f_i) = \alpha(f_i) + \beta(f_i)$ (this is how addition in $\text{End}_R(F)$ is defined). So $(\alpha + \beta)(f_i) = \sum_{j=1}^n a_{ji}f_j + \sum_{j=1}^n b_{ji}f_j = \sum_{j=1}^n (a_{ji} + b_{ji})f_j$, and therefore $[\alpha + \beta] = [\alpha] + [\beta]$.

(b) Using the result from (a), $[\alpha]$ invertible iff α an isomorphism iff $\{\alpha(f_1), \dots, \alpha(f_n)\}$ is a basis.