620-321 Algebra, Semester 1, 2009

Answers to Problem Sheet 6

2. Consider the homomorphism
$$\varphi : M_1 \oplus M_2 \to \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$$
defined by $(m_1, m_2) \mapsto (m_1 + N_1, m_2 + N_2)$. The image of (m_1, m_2) is 0 if and only if each $m_i \in N_i$. Hence the kernel is $N_1 \oplus N_2$ identified as a submodule of $M_1 \oplus M_2$. Then apply the first isomorphism theorem, after noting that the homomorphism is surjective.

3. Note that the d_i are uniquely determined up to associates, but X^{-1} and Y are not.
(a)
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -2 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
(b)
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & (1-2x-x^2) & (-1+x+x^2) \end{pmatrix} A \begin{pmatrix} 0 & -1 & 1-3x \\ 0 & 0 & 1 \\ 1 & 1+x & -1+3x^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (2-4x+3x^2+3x^3) \end{pmatrix}$$

5. In case $n = 0$, we must show $\mathbb{Z}$ is not the direct sum of two non-trivial abelian groups. So suppose $\mathbb{Z} = A \oplus B$, and $0 \neq a \in A, 0 \neq b \in B$. Then $0 \neq ab \in A \cap B$ which contradicts the property $A \cap B = 0$ of an internal direct sum.

   In case $n > 1$, we must show $\mathbb{Z}/(p^n)$ is not the direct sum of two non-trivial abelian groups. So suppose $\mathbb{Z}/(p^n) = A \oplus B$. Now $|A| = p^k$ and $|B| = p^m$ (Lagrange's theorem) where $k + m = n$ and $k > 0, m > 0$ and say $k \geq m$. Then p^k annihilates both A and B and hence $\mathbb{Z}/(p^n)$, which is a contradiction.

6. To be invertible, its invariant form must have only units down the diagonal, and hence be equivalent to the identity matrix.

7. Since R is a PID, $1 = a_1 r_1 + \cdots + a_s r_s$ for some $a_i \in R$. Let $U \leq V$ be the submodule generated by f. Then V/U has presentation matrix $[r_1, \ldots, r_s]^T$ (i.e., the matrix of the inclusion of U into V with respect to the bases $\{f\}$ and $\{f_1, \ldots, f_s\}$ for U and V respectively). This matrix is equivalent by row operations using the a_i alone to $[1, 0, \ldots, 0]^T$. The element f is the first element in the corresponding basis for V.