
620-321 Algebra, Semester 1, 2009

Assignment 2 – Solutions

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1. *Let R be a commutative ring with 1 and let I and J be ideals of R. Prove that $R/I \cong R/J$ as R-modules if and only if $I = J$. Give an example to show that it is possible to have $R/I \cong R/J$ as rings even if $I \neq J$.*

   Solution:—

   Suppose that $R/I \cong R/J$. We will show that $I \subseteq J$. The opposite inclusion follows from the same argument with the roles of I and J interchanged. Let $\varphi : R/I \to R/J$ be any isomorphism (of modules).

$$\begin{aligned}
i \in I &\implies i(r+I) = ir + I = 0 + I \quad \forall r \in R && \text{since } I \text{ is an ideal}\\
&\implies \varphi(i(r+I)) = 0 + J \quad \forall r \in R && \text{since } \varphi \text{ is a module homomorphism}\\
&\implies i\varphi(r+I) = 0 + J \quad \forall r \in R && \text{since } \varphi \text{ is a module homomorphism}\\
&\implies i(s+J) = 0 + J \quad \forall s \in R && \text{since } \varphi \text{ is surjective}\\
&\implies i(1+J) = 0 + J\\
&\implies i + J = 0 + J\\
&\implies i \in J
\end{aligned}$$

   We have shown that $I \subseteq J$. By the comment above we have $I = J$ as required.

   The converse, $I = J \implies R/I \cong R/J$, is immediate.

   As a counterexample we can take $R = \mathbb{R}[x]$, $I = (x-1)$ and $J = (x+1)$, since

$$\mathbb{R}[x]/(x-1) \cong \mathbb{R} \cong \mathbb{R}[x]/(x+1) \qquad \text{(as rings)}$$

   and $I \neq J$ because $x - 1$ and $x - 2$ are not associates.

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2. *Let $R \subseteq \mathbb{R}[x]$ be the subring consisting of all polynomials in which the coefficient of x is zero:*

$$R = \{f \in \mathbb{R}[x] \mid f(x) = a_0 + a_2x^2 + \cdots + a_nx^n, \text{ for some } a_0, a_2, \ldots, a_n \in \mathbb{R}\}$$

   Show that $\mathbb{R}[x]$ is finitely generated as an R-module, and that it is torsion-free but not free.

   Solution:—

   The set $\{1, x\} \subseteq \mathbb{R}[x]$ is a finite generating set for $\mathbb{R}[x]$ over R, since

$$\sum_{i=0}^{n} \alpha_i x^i = (\alpha_0 + \sum_{i=2}^{n} \alpha_i x^i)1 + \alpha_1 x$$

   noting that $(\alpha_0 + \sum_{i=2}^{n} \alpha_i x^i) \in R$ and $\alpha_1 \in R$.

   That ${}_R\mathbb{R}[x]$ is torsion-free follows from the fact that the ring $\mathbb{R}[x]$ has no zero divisors. More explicitly:

$$\begin{aligned}
f \in {}_R\mathbb{R}[x] \text{ is a torsion element} &\implies \exists\, g \in R - \{0\} \text{ such that } gf = 0\\
&\implies f = 0 \qquad \text{since otherwise } \deg(fg) > \deg(g) > 0
\end{aligned}$$

   To show that ${}_R\mathbb{R}[x]$ is not free, let $f, g \in \mathbb{R}[x]$ be any two elements. Then $x^2f, x^2g \in R$ and $(x^2g)f - (x^2f)g = 0$ show that the set $\{f, g\}$ is linearly dependent over R. Any basis for ${}_R\mathbb{R}[x]$

would therefore contain not more than one element. Suppose then that $\{f\}$ is a basis for ${}_R\mathbb{R}[x]$. Then

$$\begin{aligned} 1 \in {}_R\mathbb{R}[x] &\implies 1 = rf && \text{for some } r \in R \\ &\implies \deg(f) = 0 \\ &\implies (f) = R \end{aligned}$$

which is a contradiction as f is supposed to generate $\mathbb{R}[x]$.

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3. *(a) An R-module M is said to be* irreducible *if $\{0\}$ and M are the only submodules of M. Show that a torsion module M over a PID R is irreducible if and only if $M = Rm$ with* $\operatorname{ann}_R(m) = (p)$ *for some prime $p \in R$.*

   (b) M is said to be indecomposable *if it is not a direct sum of two non-zero submodules. Show that, if M is a finitely generated R-module over a PID , then M is indecomposable if and only if $M = Rm$ where* $\operatorname{ann}_R(m) = \{0\}$ *or* $\operatorname{ann}_R(m) = (p^e)$ *with p prime.*

   Solution:—

   (a) Let M be torsion module over a PID R. Let $m \in M$ be any non-zero element, and consider the homomorphism $\varphi : R \to M$ given by $\varphi(r) = rm$.

   Suppose that M is irreducible. The submodule $\varphi(R) \subseteq M$ must therefore be equal to M as $m = 1.m \in \varphi(R)$. By the first isomorphism theorem we have $\varphi(R) \cong R/\ker(\varphi)$. The kernel is $\ker(\varphi) = \operatorname{ann}_R(m)$ and is an ideal in R. It is non-trivial because m is a torsion element. As R is a PID, we have $\operatorname{ann}_R(m) = (d)$ for some nonzero $d \in R$. Hence, we have $M \cong R/(d)$. Suppose that $d = pq$. Then $\varphi((p))$ is a submodule of M. If $\varphi((p)) = 0$, then $p \in \ker(\varphi)$, and so $d|p$. In this case we would have $d \sim p$. If $\varphi((p)) = M$, then $(p) = M$ and p must be a unit. In summary, we have shown that $M = \varphi(R) = Rm$ and that $\operatorname{ann}_R(m) = (d)$ with d irreducible (and therefore prime).

   For the converse, suppose that $M = Rm$ with $\operatorname{ann}_R(m) = (p)$ for some prime $p \in R$, and consider the homomorphism φ as above. Let N be a submodule of M. Then $\varphi^{-1}(N)$ is an ideal in R (by the correspondence theorems from lectures) that contains (p). Since p is prime, we know that (p) is a maximal ideal. It follows that either $\varphi^{-1}(N) = (p)$ or $\varphi^{-1}(N) = R$. If $\varphi^{-1}(N) = (p)$, then $N = \varphi((p)) = \{0\}$. If $\varphi^{-1}(N) = R$, then $N\varphi(R) = M$. It follows that M is irreducible.

   (b) As M is finitely generated and R is a PID, we can apply the structure theorem. The primary decomposition

   $$M \cong \frac{R}{(p_1^{e_1})} \oplus \cdots \oplus \frac{R}{(p_n^{e_n})} \oplus R \oplus \cdots \oplus R$$

   is unique up to re-ordering of the summands.

   If M is indecomposable, there must only be one summand in the primary decomposition. Therefore $M \cong \frac{R}{(p^e)}$ (which implies that $M = Rm$ with $\operatorname{ann}_R(m) = (p^e)$) or $M \cong R$ (which implies that $M = Rm$ with $\operatorname{ann}_R(m) = \{0\}$).

   Conversely, if $M = Rm$ with $\operatorname{ann}_R(m) = \{0\}$ or $\operatorname{ann}_R(m) = (p^e)$ with p prime, then we have $M \cong \frac{R}{(p^e)}$ or $M \cong R$. The primary decomposition therefore has exactly one summand. If we had $M \cong A \oplus B$ for two non-trivial submodules $A, B \subseteq M$, then by considering the primary decompositions of A and B we would obtain a primary decomposition of M having at least two summands. Contradiction. Note that we can apply the structure theorem to A and B because any submodule of a finitely generated module over a PID is itself finitely generated.

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4. *Consider the matrix*

$$A = \begin{bmatrix} 2 & 1+i & 1-i \\ 8+6i & -4 & 0 \end{bmatrix} \in M_{2\times 3}(\mathbb{Z}[i])$$

over the Gaussian integers $\mathbb{Z}[i]$. *Find square matrices* X *and* Y *over* $\mathbb{Z}[i]$ *such that* XAY *is the invariant factor matrix for* A *(over* $\mathbb{Z}[i]$*).*

Solution:—

Applying row and column operations to A we obtain the following:

$$\begin{aligned}
\begin{bmatrix} 2 & 1+i & 1-i \\ 8+6i & -4 & 0 \end{bmatrix} &\sim \begin{bmatrix} 1+i & 2 & 1-i \\ -4 & 8+6i & 0 \end{bmatrix} && C_1 \leftrightarrow C_2 \\
&\sim \begin{bmatrix} 1+i & 0 & 0 \\ -4 & 12+2i & -4i \end{bmatrix} && \begin{gathered} C_2 \mapsto C_2 - (1-i)C_1 \\ C_3 \mapsto C_3 + iC_1 \end{gathered} \\
&\sim \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 12+2i & -4i \end{bmatrix} && R_2 \mapsto R_2 + (2-2i)R_1 \\
&\sim \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -4i & 12+2i \end{bmatrix} && C_2 \leftrightarrow C_3 \\
&\sim \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -4i & 2i \end{bmatrix} && C_3 \mapsto C_3 - 3iC_2 \\
&\sim \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 2i & -4i \end{bmatrix} && C_2 \leftrightarrow C_3 \\
&\sim \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 2i & 0 \end{bmatrix} && C_3 \mapsto C_3 + 2C_2
\end{aligned}$$

This is the invariant factor matrix of A. Note that $2i = (1+i)^2$. To find X we apply the row operations above to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and to find Y we apply the column operations to $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. This gives

$$X = \begin{bmatrix} 1 & 0 \\ 2-2i & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2+i & 4+3i \\ 0 & -3i & 1-6i \end{bmatrix}$$

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5. *Let* M *be the abelian group generated by* x, y, *and* z *subject to the relations:* $2x - 4y + 2z = 0$, $-2x + 10y + 4z = 0$ *and* $6x + 18z = 0$. *Determine the invariant factor decomposition of* M. *What is the torsion-free rank of* M*?*

Solution:—

Considering the corresponding matrix we have:

$$\begin{bmatrix} 2 & -2 & 6 \\ -4 & 10 & 0 \\ 2 & 4 & 18 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 6 \\ 0 & 6 & 12 \\ 0 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 12 \\ 0 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 12 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that $M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}$ and the torsion-free rank of M is 1.

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6. *List, up to isomorphism, all abelian groups of order 936, and give the annihilator of each (considering each group as a $\mathbb{Z}$-module).*

   Solution:—

   Noting that $936 = 2^3 \times 3^2 \times 13$ and considering the possible primary decompositions we obtain the following:

$\boldsymbol{G}$	**ann** $(\boldsymbol{G})$
$\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{13}$	(936)
$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{13}$	(468)
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{13}$	(234)
$\mathbb{Z}_{2^3} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{13}$	(312)
$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{13}$	(156)
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{13}$	(78)

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7. *Find a basis for the submodule of (the $\mathbb{Z}$-module) $\mathbb{Z}^3$ that is generated by the elements* $(1,0,-1)$, $(2,-3,1)$, $(0,3,1)$ *and* $(3,1,5)$.

   Solution:—

   Let N be the submodule generated by the four elements. Considering the corresponding matrix we have:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -3 & 3 & 1 \\ -1 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -3 & 3 & 1 \\ 0 & 3 & 1 & 8 \end{bmatrix} \text{(R3+R1)} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 1 \\ 0 & 3 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ 0 & 8 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & -23 & 27 \end{bmatrix} \text{(R3-8R2)} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -23 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -23 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = D$$

Only the row operations have been recorded as these are all we shall need. We know that there is a basis $\{f_1, f_2, f_3\}$ of $\mathbb{Z}^3$ such that $\{f_1, f_2, f_3\}$ is a basis for N. It follows that $N = \mathbb{Z}^3$ so we can take $\{(1,0,0),(0,1,0),(0,0,1)\}$ as our basis. Alternatively, if we continue with the general method we consider the matrix X coming from the equation $D = X^{-1}AY$. It is the columns of X that give us the basis $\{f_1, f_2, f_3\}$ of $\mathbb{Z}^3$. The matrix X is obtained from $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ by reversing the row operations in the above calculation. This gives

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 8 & 1 \end{bmatrix}$$

and we can take $\{(1,0,-1),(0,1,8),(0,0,1)\}$ as a basis of N.

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8. *Let* $R = \mathbb{R}[x]$ *and suppose that* M *is a direct sum of cyclic* R*-modules with annihilators* $(x-1)^3$*,* $(x^2+1)^2$*,* $(x-1)(x^2+1)^4$ *and* $(x+2)(x^2+1)^2$*. Determine the invariant factor decomposition of* M *and the primary decomposition of* M*.*

   Solution:—

   We have

$$\begin{aligned} M &\cong \frac{R}{((x-1)^3)} \oplus \frac{R}{((x^2+1)^2)} \oplus \frac{R}{((x-1)(x^2+1)^4)} \oplus \frac{R}{((x+2)(x^2+1)^2)} \\ &\cong \frac{R}{((x-1)^3)} \oplus \frac{R}{((x^2+1)^2)} \oplus \frac{R}{(x-1)} \oplus \frac{R}{((x^2+1)^4)} \oplus \frac{R}{(x+2)} \oplus \frac{R}{((x^2+1)^2)} \\ &\cong \frac{R}{(x-1)} \oplus \frac{R}{((x-1)^3)} \oplus \frac{R}{(x+2)} \oplus \frac{R}{((x^2+1)^2)} \oplus \frac{R}{((x^2+1)^2)} \oplus \frac{R}{((x^2+1)^4)} && \text{(primary)} \\ &\cong \frac{R}{((x^2+1)^2)} \oplus \frac{R}{((x-1)(x^2+1)^2)} \oplus \frac{R}{((x-1)^3(x+2)(x^2+1)^4)} && \text{(invariant factor)} \end{aligned}$$

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9. *Suppose that* $T : \mathbb{C}^6 \to \mathbb{C}^6$ *is a linear transformation having minimal polynomial*

$$\mu_T(t) = (t-5)^2(t-6)^2.$$

   Give all possible Jordan Normal Form matrices for T *(up to re-ordering of the elementary Jordan blocks).*

   Solution:—

   The Jordan Normal Form will be a 6×6 matrix having Jordan elementary block of size at most 2×2. The possibilities are:

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

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10. *Calculate the Jordan Normal Form for the following matrix A by calculating the invariant factor matrix of $xI - A \in M_4(\mathbb{C}[x])$.*

$$A = \begin{bmatrix} 3 & -1 & 1 & 7 \\ 9 & -3 & -7 & -1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

What are the characteristic and minimal polynomials of A?

Solution:—

$$xI - A = \begin{bmatrix} x-3 & 1 & -1 & -7 \\ -9 & x+3 & 7 & 1 \\ 0 & 0 & x-4 & 8 \\ 0 & 0 & -2 & x+4 \end{bmatrix} \sim \begin{bmatrix} 1 & x-3 & -1 & -7 \\ x+3 & -9 & 7 & 1 \\ 0 & 0 & x-4 & 8 \\ 0 & 0 & -2 & x+4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ x+3 & -x^2 & x+10 & 7x+22 \\ 0 & 0 & x-4 & 8 \\ 0 & 0 & -2 & x+4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -x^2 & x+10 & 7x+22 \\ 0 & 0 & x-4 & 8 \\ 0 & 0 & -2 & x+4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & x+4 \\ 0 & 0 & x-4 & 8 \\ 0 & -x^2 & x+10 & 7x+22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & x+4 \\ 0 & x-4 & 0 & 8 \\ 0 & x+10 & -x^2 & 7x+22 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & x+4 \\ 0 & 0 & 0 & \frac{1}{2}x^2 \\ 0 & 0 & -x^2 & \frac{1}{2}x^2+14x+42 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}x^2 \\ 0 & 0 & -x^2 & \frac{1}{2}x^2+14x+42 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x^2 \\ 0 & 0 & -x^2 & 14x+42 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{14}x^3 & -3x \\ 0 & 0 & -x^2 & 14x+42 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{14}x^3 & -3x \\ 0 & 0 & \frac{1}{3}x^3-x^2 & 42 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 42 & \frac{1}{3}x^3-x^2 \\ 0 & 0 & -3x & \frac{1}{14}x^3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 42 & \frac{1}{3}x^3-x^2 \\ 0 & 0 & 0 & \frac{1}{42}x^4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 42 & 0 \\ 0 & 0 & 0 & \frac{1}{42}x^4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x^4 \end{bmatrix}$$

This is the invariant factor matrix. The corresponding decomposition is ${}_{\mathbb{C}[x]}\mathbb{C}^4 \cong \frac{\mathbb{C}[x]}{(x^4)}$. The Jordan Normal Form of the matrix A is therefore:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic and minimal polynomials are $\chi_A(t) = t^4$ and $\mu_A(t) = t^4$.
