620-321 Algebra, Semester 1, 2009

Assignment 1

Due: Monday 6th April, 4pm

Late submissions will ***not*** be accepted. A completed plagiarism cover sheet should be submitted with the assignment. All answers should be appropriately justified.

1. Give examples for each of the following (you need not prove the required properties):
   (a) A non-commutative ring with no multiplicative identity.
   (b) A non-commutative ring with multiplicative identity.
   (c) A commutive ring with no multiplicative identity.
   (d) A commutive ring with multiplicative identity that is not an integral domain.
   (e) An integral domain that is not a UFD.
   (f) A ring that is not a PID, but in which all ideals are principal.

2. Define new operations of $\mathbb{Z}$ by: $x \oplus y = x + y + 1$ and $x \odot y = xy + x + y$.
   (a) Prove that with respect to these operations $\mathbb{Z}$ forms a ring.
   (b) Prove that $(\mathbb{Z}, \oplus, \odot)$ is isomorphic to $(\mathbb{Z}, +, \times)$ (i.e., $\mathbb{Z}$ with the usual operations).

3. Let $a, b \in R$ be two elements in a commutative ring R. Show carefully that
$$R/(a, b) \cong (R/(a))/(b + (a))$$

4. Let R be a commutative ring with multiplicative identity, let I be an ideal in R. Let
$$J = \{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in I\}$$
Show that J is an ideal in R. (J is called the *radical* of I.)

5. Let R be a commutative ring with 1, and suppose that every element $x \in R$ satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in R is maximal.

6. Let R be a UFD. Let $a, b \in R$ be two elements which are not both zero. An element $m \in R$ is a *least common multiple* of a and b (denoted $\mathrm{lcm}(a, b)$) if it satisfies:
   1: $a|m$ and $b|m$;
   2: if $m' \in R$ is such that $a|m'$ and $b|m'$, then $m|m'$.

   (a) Prove that $\mathrm{lcm}(a, b)$ exists and is unique up to multiplication by a unit.
   (b) Let m be a lcm of a and b, and let d be a gcd of a and b. Prove that md is an associate of ab.

7. Let R be the following subset of $\mathbb{Q}$:
$$R = \{x \in \mathbb{Q} \mid x = \frac{a}{b}, \text{ for some } a, b \in \mathbb{Z}, \text{ with } b \text{ not divisible by } 3\}.$$
   (a) Show that R is a subring of $\mathbb{Q}$.
   (b) Describe the units of R.
   (c) Show that every proper ideal in R has the form (3^k) (i.e., the ideal in R generated by 3^k) for some positive $k \in \mathbb{Z}$ (depending on the ideal).
   (d) Show that $R/(3)$ is a field.

8. (a) Show that the following are irreducible in $\mathbb{Q}[x]$:
   i. $x^2 + x + 1$ ii. $x^5 + 7x + 7$ iii. $x^5 + 3x + 2$
   (b) Factorise the polynomial $x^3 + x + 1$ into irreducible factors in:
   i. $\mathbb{Z}_2[x]$ ii. $\mathbb{Z}_3[x]$

9. Show that $\mathbb{Z}[\zeta]$ is a Euclidean Domain, where $\zeta = e^{2\pi i/3}$.

10. Use the Euclidean algorithm to find a gcd in $\mathbb{Q}[x]$ of
$$f(x) = x^4 - x^3 + 2x^2 - x + 1 \quad \text{and} \quad g(x) = x^4 + x^3 + 2x^2 + x + 1$$
and write it as a $\mathbb{Q}[x]$-linear combination of f and g.

The University of Melbourne

DEPARTMENT OF MATHEMATICS AND STATISTICS

PLAGIARISM DECLARATION SHEET: Semester 1, 2009

Subject number and name:____________________________________

Tutor:____

Tutorial time and place:____________________________________

Name (CAPITALS):____

Student Number:____________________

PLAGIARISM is the presentation by a student of an assignment or piece of work which has in fact been copied in whole or in part from another student's work, or from any other source (eg published books or periodicals or material from Internet sites), without due acknowledgement in the text.

TEAMWORK: It is the goal of all courses in the Department of Mathematics and Statistics to develop generic skills that will be useful in the workplace in addition to developing specific skills, knowledge and understanding in Mathematics and Statistics. A key generic skill is teamwork. Students are encouraged to work with each other (both inside and outside tutorials) to develop their technical skills and to increase their knowledge and understanding of the curriculum, in part because this develops the capacity for teamwork.

In the case of marked assignments or other non-examination assessments in the Department of Mathematics and Statistics, such teamwork includes general discussion and sharing of ideas on the work. All written work must however (without specific authorisation to the contrary) be done by individual students. Students are neither permitted to copy any part of another student's work nor permitted to allow their own work to be copied by other students.

DECLARATION:

- *I declare that all work submitted for assessment in this subject will be my own work and does not involve plagiarism or teamwork other than that authorised in the general terms above or that authorised for any particular piece of work.*
- *I understand that this declaration covers all work submitted for assessment for Semester 1, 2009, in this subject.*

Signed:____

Date:__________________________