620-321 Algebra, Semester 1, 2009

# Assignment 2

Due: Monday 18th May, 4pm

Late submissions will ***not*** be accepted. All answers should be appropriately justified.

1. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$. Prove that $R/I \cong R/J$ as $R$-modules if and only if $I = J$. Give an example to show that it is possible to have $R/I \cong R/J$ as rings even if $I \neq J$.

2. Let $R \subseteq \mathbb{R}[x]$ be the subring consisting of all polynomials in which the coefficient of $x$ is zero:
$$R = \{f \in \mathbb{R}[x] \mid f(x) = a_0 + a_2x^2 + \cdots + a_nx^n, \text{ for some } a_0, a_2, \ldots, a_n \in \mathbb{R}\}$$
Show that $\mathbb{R}[x]$ is finitely generated as an $R$-module, and that it is torsion-free but not free.

3. (a) An $R$-module $M$ is said to be *irreducible* if $\{0\}$ and $M$ are the only submodules of $M$. Show that a torsion module $M$ over a PID $R$ is irreducible if and only if $M = Rm$ with $\operatorname{ann}_R(m) = (p)$ for some prime $p \in R$.

   (b) $M$ is said to be *indecomposable* if it is not a direct sum of two non-zero submodules. Show that, if $M$ is a finitely generated $R$-module over a PID , then $M$ is indecomposable if and only if $M = Rm$ where $\operatorname{ann}_R(m) = \{0\}$ or $\operatorname{ann}_R(m) = (p^e)$ with $p$ prime.

4. Consider the matrix
$$A = \begin{bmatrix} 2 & 1+i & 1-i \\ 8+6i & -4 & 0 \end{bmatrix} \in M_{2\times 3}(\mathbb{Z}[i])$$
over the Gaussian integers $\mathbb{Z}[i]$. Find square matrices $X$ and $Y$ over $\mathbb{Z}[i]$ such that $XAY$ is the invariant factor matrix for $A$ (over $\mathbb{Z}[i]$).

5. Let $M$ be the abelian group generated by $x$, $y$, and $z$ subject to the relations: $2x - 4y + 2z = 0$, $-2x + 10y + 4z = 0$ and $6x + 18z = 0$. Determine the invariant factor decomposition of $M$. What is the torsion-free rank of $M$?

6. List, up to isomorphism, all abelian groups of order 936, and give the annihilator of each (considering each group as a $\mathbb{Z}$-module).

7. Find a basis for the submodule of (the $\mathbb{Z}$-module) $\mathbb{Z}^3$ that is generated by the elements $(1, 0, -1)$, $(2, -3, 1)$, $(0, 3, 1)$ and $(3, 1, 5)$.

8. Let $R = \mathbb{R}[x]$ and suppose that $M$ is a direct sum of cyclic $R$-modules with annihilators $(x-1)^3$, $(x^2+1)^2$, $(x-1)(x^2+1)^4$ and $(x+2)(x^2+1)^2$. Determine the invariant factor decomposition of $M$ and the primary decomposition of $M$.

9. Suppose that $T : \mathbb{C}^6 \to \mathbb{C}^6$ is a linear transformation having minimal polynomial
$$\mu_T(t) = (t-5)^2(t-6)^2.$$
Give all possible Jordan Normal Form matrices for $T$ (up to re-ordering of the elementary Jordan blocks).

10. Calculate the Jordan Normal Form for the following matrix $A$ by calculating the invariant factor matrix of $xI - A \in M_4(\mathbb{C}[x])$.
$$A = \begin{bmatrix} 3 & -1 & 1 & 7 \\ 9 & -3 & -7 & -1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$
What are the characteristic and minimal polynomials of $A$?