620-321 Algebra, Semester 1, 2009

Problem Sheet 1

1. Let R be an integral domain such that $x^2 = x$ for all $x \in R$. Show that either $R = \{0\}$ or R has exactly two elements.

2. List all units in the following rings:
   (a) $\mathbb{Z}$ (b) $\mathbb{Z} \times \mathbb{Z}$ (c) $\mathbb{Z}_5$ (d) $\mathbb{Z}_4$ (e) $\mathbb{Q}$ (f) $\mathbb{Z}[i]$ (g) $\mathbb{R}[x]$

3. True or false:
   (a) Every field is also a ring.
   (b) Every ring with a multiplicative identity has at least two elements.
   (c) Every subring of a field is a subfield.
   (d) The non-zero elements in a field form a group under multiplication.
   (e) Addition in a ring is always commutative.
   (f) As a ring, $\mathbb{Z}$ is isomorphic to $n\mathbb{Z}$ for all $n \geq 1$.
   (g) The direct product of two integral domains is again an integral domain.

4. Give the multiplication table for the multiplicative group of units in $\mathbb{Z}_{12}$. To which group of order 4 is it isomorphic?

5. Let R be a commutative ring with multiplicative identity. Show that R is a field if and only if R contains exactly two ideals ($\{0\}$ and R).

6. Give an example of a ring R with multiplicative identity 1_R that has a proper subring $S \leqslant R$ with multiplicative identity $1_S \neq 1_R$.

7. Show that in a ring there is at most one multiplicative identity.

8. Show that in a ring R we have:
   (a) $0a = a0 = 0 \quad \forall a \in R$
   (b) $a(-b) = (-a)b = -(ab) \quad \forall a, b \in R$
   (c) $(-a)(-b) = ab \quad \forall a, b \in R$

9. Prove the cancellation law.

10. The direct product $R \times S$ of two rings is given by the set $\{(r, s) \mid r \in R, s \in S\}$ with operations given by

$$(r_1, s_1) + (r_2, s_2) = (r_1 +_R r_2, s_1 +_S s_2)$$
$$(r_1, s_1) \times (r_2, s_2) = (r_1 \times_R r_2, s_1 \times_S s_2)$$

   (a) Is the map $r \mapsto (r, 0)$ from R to $R \times S$ a ring homomorphism ?
   (b) What about $r \mapsto (r, r)$ from R to $R \times R$?

11. Denote the ring $\mathbb{Z}/(n)$ by $\mathbb{Z}_n$.
(a) Is $\mathbb{Z}_8$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$? (b) Is $\mathbb{Z}_{15}$ isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_5$?

12. If I, J are ideals in R, the sum of I and J denoted $I + J$ is defined by

$$I + J = \{r \in R \mid r = x + y, x \in I, y \in J\}.$$

Show that $I + J$ is again an ideal in R. Show that if $I + J = R$, then $R/(I \cap J) \cong R/I \times R/J$.

13. Using the above exercise or otherwise, show that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ if and only if the $\gcd(m, n) = 1$.

14. Let R be a commutative ring with 1. Show that if I is an ideal in R, then $I = R$ if and only if I contains a unit.