620-321 Algebra, Semester 1, 2009

Problem Sheet 10

1. Show that $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. Give two isomorphisms $\mathbb{R}[x]/(x^2+1) \to \mathbb{C}$. Identify the group $G(\mathbb{C}/\mathbb{R})$.
2. Determine the degree of the splitting fields of the following elements of $\mathbb{Q}[x]$.
   (a) x^4-1 (b) x^3-2 (c) x^4+1
3. Determine all $\mathbb{Q}$-automorphisms of the field $\mathbb{Q}(\sqrt[3]{2})$.
4. Prove that the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\sqrt{2})$ sending $\sqrt{2}$ to $-\sqrt{2}$ is not continuous.
5. Show that $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is a Galois extension of $\mathbb{Q}$ and identify its Galois group.
6. Explain why it is impossible to construct a regular 9-gon using straight-edge and compass.
7. Let K be an extension field of F such that $[K : F]$ is prime. If u is an element of $K \setminus F$ show that $F(u) = K$.
8. Let u, v be algebraic over F with minimal polynomials $p(x), q(x)$ respectively. If the degrees of $p(x)$ and $q(x)$ are co-prime, show that:
   $$[F(u,v) : F] = \deg(p(x)) \times \deg(q(x)).$$
9. If K is an extension of $\mathbb{Q}$ with $[K : \mathbb{Q}] = 2$, show that $K = \mathbb{Q}(\sqrt{d})$ for some (square-free) integer d.
10. Show that x^2-3 and x^2-2x-2 are irreducible in $\mathbb{Q}[x]$ and have the same splitting field.
11. Find the dimensions of the splitting fields over $\mathbb{Q}$ of
    (a) x^3-56; and
    (b) x^4-4x^2-5.
12. Find the dimension of a splitting field of x^3+x+1 over $\mathbb{Z}_2$.
13. Show that $\mathbb{Q}(5^{\frac{1}{3}})$ cannot be the splitting field of any polynomial over $\mathbb{Q}$. What is the Galois group of $\mathbb{Q}(5^{\frac{1}{3}})$ over $\mathbb{Q}$?