620-321 Algebra, Semester 1, 2009

Problem Sheet 3

1. Let $\varphi : R \to S$ be a homomorphism, and define a map $\Phi : R[x] \to S[x]$ by

$$\Phi(a_0 + a_1 x + \cdots + a_n x^n) = \varphi(a_0) + \varphi(a_1)x + \cdots + \varphi(a_n)x^n$$

   Show that Φ is a homomorphism.

2. True or false:
   (a) Every field is a UFD.
   (b) Every field is a PID.
   (c) Every PID is a UFD.
   (d) Every UFD is a PID.
   (e) In a UFD, any two irreducibles are associates.
   (f) If D is a PID, then $D[x]$ is a PID.
   (g) If D is a UFD, then $D[x]$ is a UFD.
   (h) Irreducible elements in an integral domain are prime.
   (i) In a UFD, if p is irreducible and $p|a$, then p appears in every factorisation of a.

3. Express the following as the product of a constant polynomial and a primitive polynomial:
   (a) $18x^2 - 12x + 48$ in $\mathbb{Z}[x]$;
   (b) $18x^2 - 12x + 48$ in $\mathbb{Q}[x]$;
   (c) $2x^2 - 3x + 6$ in $\mathbb{Z}_7[x]$.

4. Factor $4x^2 - 4x + 8$ into a product of irreducibles in: (a) $\mathbb{Z}[x]$; (b) $\mathbb{Q}[x]$; (c)$\mathbb{Z}_{11}[x]$.

5. Let D be an integral domain, and p, $q \in D$ with $q|p$. Show that:
   (a) If p is a unit, then q is a unit.
   (b) If p is irreducible, then either q is a unit or p and q are associates.
   (c) If p and q are associates, then p is irreducible iff q is irreducible.

6. Show that irreducible elements in a UFD are prime.

7. Show that $R[x]/(x - a)$ is isomorphic to R for any $a \in R$. (By $(x - a)$ here we mean the ideal in $R[x]$ generated by $x - a$.)

8. Show that $\mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}[1/2]$.

9. Suppose that R is a commutative ring and $a \in R$ a fixed element. Show that the map from $R[x]$ to itself defined by

$$a_0 + a_1 x + \cdots + a_n x^n \mapsto a_0 + a_1(x - a) + \cdots + a_n(x - a)^n$$

   is an isomorphism of rings. Deduce that if $f(x) \in R[x]$, then $f(x)$ can be expressed in the form $f(x) = \sum b_i(x - a)^i$ for suitable $b_i \in R$.

10. If we regard the reals $\mathbb{R}$ as a subring of the complex numbers $\mathbb{C}$, we can extend the inclusion to a homomorphism $\varphi : \mathbb{R}[x] \to \mathbb{C}$ by defining $\varphi(x) = i \in \mathbb{C}$. Show that φ induces an isomorphism $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

11. Consider the map φ from $\mathbb{Z}[x]$ to the real numbers which is identity on $\mathbb{Z}$, and takes x to $(1+\sqrt{2})$. Show that the kernel of φ is a principal ideal and find a generator for this ideal.

12. Prove that if R is a PID, then a gcd of $a, b \in R$ is an R-linear combination of a, b.

13. Let R be a PID and let S be an integral domain containing R. Let $a, b, d \in R$. If d is a gcd of a, b in R, show that d is a gcd of a, b in S.

14. Show that in a UFD a gcd of da, db is d times a gcd of a, b.

15. If R is a PID and $0 \neq p \in R$, then the following are equivalent:

    (a) the ideal (p) is prime;

    (b) p is an irreducible element;

    (c) (p) is a maximal ideal in R;

    (d) $R/(p)$ is a field;

    (e) $R/(p)$ is an integral domain.

    This statement collects the results of several earlier exercises and facts proved in lectures. For this exercise you should write out a proof of these implications in the indicated order: each implies the next and the last implies the first. Note that this result applies to the case $R = F[x]$ where F is a field and $p = p(x)$ is a non-constant polynomial.