620-321 Algebra, Semester 1, 2009

Problem Sheet 4

1. Let R be an integral domain. If $f, g \in R[x]$ and if the leading coefficient of g is a unit, show that $\exists\ q, r \in R[x]$ such that
(a) $f(x) = g(x)q(x) + r(x)$, and
(b) either $r(x) = 0$ or $\deg(r) < \deg(g)$.

2. Find a gcd of $x^3 - 6x^2 + x + 4$ and $x^5 - 6x + 1$ in $\mathbb{Q}[x]$.

3. Consider the polynomials $f(x) = x^3 - 6x^2 + x + 4$ and $g(x) = x^4 - 6x^3 + 5$ in $\mathbb{Q}[x]$. Find a gcd d of f and g and then find polynomials a and b in $\mathbb{Q}[x]$ such that $d = af + bg$.

4. Use the Euclidean algorithm to calculate $\gcd(x^3 + 2x^2 + 4x - 7, x^2 + x - 2)$ in $\mathbb{Q}[x]$, and express it as a linear combination of the two polynomials.

5. Show that if $R[x]$ is a PID, then R is a field. (This is the converse of a result from the lectures.)

6. Show that every field is a ED.

7. Show that the following are irreducible in $\mathbb{Q}[x]$:
(a) $x^2 - 12$ (b) $8x^3 + 6x^2 - 9x + 24$ (c) $2x^{10} - 25x^3 + 10x^2 - 30$

8. Which are fields? (a) $\mathbb{Q}[x]/(x^2 - 5x + 6)$ (b) $\mathbb{Q}[x]/(x^2 - 6x + 6)$

9. Factor $x^5 + 5x + 5$ into irreducible factors in $\mathbb{Q}[x]$ and in $\mathbb{Z}_2[x]$.

10. By considering the image of each polynomial in $\mathbb{Z}_2[x]$, show that the following are irreducible in $\mathbb{Q}[x]$.
(a) $x^2 + 2345x + 125$ (b) $x^3 + 5x^2 + 10x + 5$ (c) $x^5 + x^4 - 4x^3 + 2x^2 + 4x + 1$

11. If p, q are relatively prime in $\mathbb{Z}$, show that a gcd of p, q in $\mathbb{Z}[i]$ is 1.

12. Factor the following into irreducibles in $\mathbb{Z}[i]$: (a) 5 (b) 7 (c) 4+3i

13. (Rational Root Test) If the reduced fraction $\frac{r}{s}$ is a root of $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ show that $r|a_0$ and $s|a_n$. Deduce that if $f(x)$ is monic and has a rational root, then it has an integral root.

14. Determine which of the following is irreducible in $\mathbb{Q}[x]$:
(a) $x^4 - 16x^2 + 4$ (b) $x^4 - 32x^2 + 4$.

15. Test for irreducibility the following polynomials $\mathbb{Q}[x]$:
(a) $x^4 - x^3 - x^2 - x - 2$
(b) $2x^4 - 5x^3 + 3x^2 + 4x - 6$
(c) $7x^3 + 6x^2 + 4x + 4$
(d) $9x^4 + 4x^3 - x + 7$.

16. Test each of the following for irreducibility in $\mathbb{Q}[x]$:
(a) $x^5 - 4x + 22$ (b) $2x^5 + 12x^4 - 15x^3 + 18x^2 - 45x + 3$ (c) $x^4 + 1$

17. (a) Show that there are irreducible polynomials of every degree in $\mathbb{Q}[x]$.
(b) Show that there are infinitely many irreducible polynomials of any given degree in $\mathbb{Q}[x]$.

18. Factorize $x^3 + x^2 + 1$ in $\mathbb{Z}_p[x]$, for $p = 2, 3$.

19. Find all monic polynomials of degree ≤ 2 in $\mathbb{Z}_3[x]$. Determine which of these polynomials are irreducible.

20. Determine all irreducible polynomials of degree ≤ 4 in $\mathbb{Z}_2[x]$.