620-321 Algebra, Semester 1, 2009

Problem Sheet 5

1. Let M be an R-module. Show that for all $r \in R$ and $m \in M$ we have:
(a) $0_R m = 0_M$ (b) $r 0_M = 0_M$ (c) $(-r)m = -(rm) = r(-m)$

2. Let F and G be two free R-modules of rank m and n respectively. Show that the R-module $F \oplus G$ is free of rank $m + n$.

3. Show that if N and M/N are finitely generated as R-modules, then M is also a finitely generated R-module.

4. Show that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module.

5. Describe the quotient modules M/N, where M, N are the following $\mathbb{Z}$-modules :

   (a) $M = \mathbb{Z} \oplus \mathbb{Z}$ and N the submodule generated by $(0, 3)$.

   (b) $M = \mathbb{Z} \oplus \mathbb{Z}$ and N the submodule generated by $(2, 0)$ and $(0, 3)$.

   (c) $M = \mathbb{Z} \oplus \mathbb{Z}$ and N the submodule generated by $(2, 3)$.

   In each case, describe the torsion submodule of M/N.

6. A module is called *cyclic* if it has a generating set with one element.

   (a) Is a quotient module of a cyclic module cyclic?

   (b) Is a submodule of a cyclic module cyclic?

7. Let V be a two dimensional vector space over $\mathbb{Q}$ having basis $\{v_1, v_2\}$. Let T be the linear transformation on V defined by $T(v_1) = 3v_1 - v_2$, $T(v_2) = 2v_2$. Make V into a $\mathbb{Q}[x]$-module by defining $x \cdot u = T(u)$.

   (a) Show that the subspace $U = \{av_2 \mid a \in \mathbb{Q}\}$ of V spanned by v_2 is actually a $\mathbb{Q}[x]$-submodule of V.

   (b) Consider the polynomial $f = x^2 + 2x - 3$. Determine the vectors $f(x) \cdot v_1$ and $f(x) \cdot v_2$, that is, express them as linear combinations of v_1 and v_2.

8. Show that the order ideal of an element is an ideal.

9. Show that a free module over an ID is torsion-free.

10. Show that R considered as a module over itself is torsion-free if and only if R is an integral domain.

11. Let F be a free R-module and fix a basis $\{f_1, \ldots, f_n\}$ for F. Recall that the matrix $[\alpha] \in M_n(R)$ of a homomorphism $\alpha : F \to F$ is given by $[\alpha] = (a_{ji})$ where $\alpha(f_i) = \sum_{j=1}^{n} a_{ji} f_j$. Denote by $End_R(F)$ the ring of all R-module homomorphisms from F to F.

    (a) Show that the map $End_R(F) \to M_n(R)$ given by $\alpha \mapsto [\alpha]$ is a ring isomorphism.

    (b) Show that $[\alpha]$ is invertible iff $\{\alpha(f_1), \ldots, \alpha(f_n)\}$ is a basis of F.