620-321 Algebra, Semester 1, 2009

Problem Sheet 6

1. Let R be a ring (commutative with 1) and V a free module of finite rank over R. Prove or disprove:
(a) Every set of generators of V contains a basis of V;
(b) Every linearly independent set in V can be extended to a basis of V.

2. Show that if $N_i \subset M_i$, $1 \leq i \leq 2$ are R- modules, then
$$\frac{M_1 \oplus M_2}{N_1 \oplus N_2} \cong \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$$

3. Given the matrix A, find invertible matrices $X^{-1}, Y \in M_3(R)$ and $d_1, d_2, d_3 \in R$ such that $X^{-1}AY = \mathrm{diag}(d_1, d_2, d_3)$ and $d_1|d_2|d_3$.
$$\text{(a) } R = \mathbb{Z}, \quad A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} \qquad \text{(b) } R = \mathbb{Q}[x], \quad A = \begin{pmatrix} 1-x & 1+x & x \\ x & 1-x & 1 \\ 1+x & 2x & 1 \end{pmatrix}$$

4. Show that $\mathbb{Q}$ as a $\mathbb{Z}$-module is torsion-free but not free.

5. Show that the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, where p is a prime and n a non-negative integer, is not a direct sum of two non-trivial $\mathbb{Z}$-modules.

6. Show that an $n \times n$ matrix over a PID is invertible if and only if it is equivalent to the identity matrix.

7. Let $f_1, f_2, ..., f_s$ be a basis of a free module V over a PID R. Suppose that $f = r_1 f_1 + r_2 f_2 + \cdots + r_s f_s$ and that 1 is a gcd of $r_1, r_2, \ldots, r_s$. Show that f is a part of a basis for V.