620-321 Algebra, Semester 1, 2009

Problem Sheet 7

1. How many abelian groups of order 136 are there? Give the primary and invariant factor decompositions of each.

2. Determine the invariant factors of the abelian group $C_{100} \oplus C_{36} \oplus C_{150}$.

3. Find a direct sum of cyclic groups which is isomorphic to the abelian group $\mathbb{Z}^3/N$, where N is generated by $\{(2,2,2),(2,2,0),(2,0,2)\}$.

4. Find the invariant factor matrices over $\mathbb{Z}$ for the first three of the following matrices, and over $\mathbb{Q}[x]$ for the last two of the following matrices:

   (a) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{bmatrix}$

   (d) $\begin{bmatrix} x & 1 & -2 \\ -3 & x+4 & -6 \\ -2 & 2 & x-3 \end{bmatrix}$ (e) $\begin{bmatrix} x & 0 & 0 \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x^2 \end{bmatrix}$

5. Let V be the $\mathbb{Z}[i]$-module generated by elements u and v with relations

   $$(1+i)u + (2-i)v = 0 \quad \text{and} \quad 3u + 5iv = 0\,.$$

   Write V as a direct sum of cyclic modules.

6. Find an isomorphic direct product of cyclic groups, where V is an abelian group generated by x, y, z and subject to relations:

   (a) $3x + 2y + 8z = 0$, $2x + 4z = 0$

   (b) $x + y = 0$, $2x = 0$, $4x + 2z = 0$, $4x + 2y + 2z = 0$

   (c) $2x + y = 0$, $x - y + 3z = 0$.

7. Suppose that the abelian group M is generated by three elements x, y, z subject to the relations $4x + y + 2z = 0, 5x + 2y + z = 0, 6y - 6z = 0$. Determine the invariant factors of M and hence exhibit M as a direct sum of cyclic groups.

8. Suppose that the linear transformation T acts on an 8 dimensional complex vector space V. Using T we make V into a $\mathbb{C}[t]$-module (where t is an indeterminant) in the usual way. Suppose that as a $\mathbb{C}[t]$-module $V \cong \mathbb{C}[t]/(t+5)^2 \oplus \mathbb{C}[t]/(t-3)^3(t+5)^3$. What is the Jordan (normal) form for the transformation T? What are the eigenvalues of T and how many eigenvectors does T have? What are the minimal and characteristic polynomials of T?

9. Let $\varphi : Z^k \to Z^k$ be a homomorphism given by multiplication by an integer matrix A. Show that the image of φ has finite index if and only if $\det A \neq 0$, and if so the index of $\varphi(Z^k)$ in Z^k is equal to $|\det A|$