

Problem Sheet 8

1. Determine the Jordan normal form of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- (a) By deducing it from the characteristic and minimal polynomials;
 (b) By calculating the invariant factor matrix of $xI - A$.
2. Find all possible Jordan normal forms for a matrix whose characteristic polynomial is $(t + 2)^2(t - 5)^3$
3. Let $R = \mathbb{Q}[x]$ and suppose that the torsion R -module M is a direct sum of four cyclic modules whose annihilators (order ideals) are $(x - 1)^3$, $(x^2 + 1)^2$, $(x - 1)(x^2 + 1)^4$, and $(x + 2)(x^2 + 1)^2$. Determine the primary components and invariant factors of M . If M is thought of as a vector space over \mathbb{Q} on which x acts as a linear transformation denoted A , determine the minimum and characteristic polynomials of A and the dimension of M over \mathbb{Q} . If the underlying vector space were \mathbb{C} instead of \mathbb{Q} , would A be diagonalizable?
4. Find $\text{irr}(a, \mathbb{Q})$ and $\text{deg}(a, \mathbb{Q})$ for
 (a) $a = \sqrt{3 - \sqrt{6}}$ (b) $a = \sqrt{\left(\frac{1}{3}\right) + \sqrt{7}}$ (c) $a = \sqrt{2} + i$
 [You should justify why the polynomial is irreducible.]
5. Find the dimension and a basis for the following extensions:
 (a) $\mathbb{R}(\sqrt{2} + i)$ over \mathbb{R} ;
 (b) $\mathbb{Q}(\sqrt{2} + i)$ over \mathbb{Q} ;
 (c) $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over \mathbb{Q} ;
 (d) $\mathbb{Q}(\sqrt{3}, i)$ over \mathbb{Q} .
6. Show that the following elements are algebraic over \mathbb{Q} and find their minimal polynomials:
 (a) $2^{\frac{1}{3}}$
 (b) $\sqrt{3} + \sqrt{2}$
 (c) $\frac{(\sqrt{5}+1)}{2}$
 (d) $\frac{(i\sqrt{3}-1)}{2}$.