620-321 Algebra, Semester 1, 2009

Problem Sheet 9

1. Show that the set of algebraic numbers (over $\mathbb{Q}$) in $\mathbb{R}$ forms a subfield of $\mathbb{R}$. (Use that $a \in \mathbb{R}$ is algebraic iff $[\mathbb{Q}(a) : \mathbb{Q}]$ is finite.)

2. (a) Show that $x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$.

   (b) Let a be a root x^2+1 in an extension field of $\mathbb{Z}_3$. List the elements of $\mathbb{Z}_3(a)$ (using the basis $\{1, a\}$ of $\mathbb{Z}_3(a)$ as a vector space over $\mathbb{Z}_3$).

   (c) Give the multiplication table for $(\mathbb{Z}_3(a))^*$. Identify the group.

3. Find a thirteenth root of 3 in $\mathbb{F}_{13}$ (a field containing exactly 13 elements).

4. Let $\mathbb{F}_4$ be a field containing 4 elements. Write out the addition and multiplication tables for $\mathbb{F}_4$.

5. If K is a finite field of characteristic $p > 0$, use the binomial formula to show the function $\varphi : K \to K$ defined by $\varphi(a) = a^p$ is a ring homomorphism. Conclude that φ is an isomorphism and hence every element of k has a p-th root in k.

6. If $f(x) \in \mathbb{Z}_p[x]$ and if u is a root of $f(x)$ in some extension of $\mathbb{Z}_p$, show that u^p is also a root of $f(x)$ in that extension.

7. If the finite field K has p^n elements, write down explicitly a polynomial in $K[x]$ which has no roots in K. Conclude that no finite field is algebraically closed.

8. If K is a finite field of order p^n, show that K has exactly one subfield of order p^d for any $d|n$.