Department of Mathematics and Statistics
620–342
Industrial and Applied Mathematics

Assignment 1. Due: September 19

This assignment counts for 15% of the marks for this subject.

Question 1

At time t, the velocity field $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ is given by

$$u(x, y, t) = \frac{x}{1+t}; \; v(x, y, t) = \frac{-y}{1+2t}$$

i. Find the streamlines in parametric form

ii. Using the result in (i) find the curves giving the shape of the streamlines in Cartesian form. What form do they take: at $t = 0$, $t \to \infty$?

iii. Find the streamfunction for this flow at $t = 0$. Using this result, calculate the streamlines in Cartesian form.

iv. Can you find the streamfunction for $t > 0$? Explain your answer.

v. Find the path of the particle that was located at (X,Y) at t=0

vi. Verify from the particle path that

$$\frac{\partial \mathbf{u}}{\partial t}\, |_{\mathbf{R}} = \frac{D\mathbf{u}}{Dt}$$

Question 2

Calculate the velocity field corresponding to the following Stokes streamfunction in spherical polar coordinates

$$\Psi = \frac{1}{2}U(r^2 - \frac{a^3}{r})\sin^2\theta$$

where U and a are positive constants. Sketch the streamlines for this flow, assuming $\sigma \geq a$. In particular, describe the flow as $r \to \infty$ and the streamline $\Psi = 0$. Also, find any stagnation points in the flow (points where $\mathbf{u} = \mathbf{0}$), calculate $\nabla \times \mathbf{u}$ and establish whether the flow is incompressible or compressible. Describe what sort of flow it could possibly represent.

Question 3

i. Prove that if b_m are components of a vector, $\omega_{ik} = \varepsilon_{ijk} b_j$ are the components of an antisymmetric second order tensor

   Hint: consider $\mathbf{a} = \mathbf{b} \times \mathbf{c}$ and use the Quotient Rule.

ii. Use the identity

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

   to show (in Cartesian coordinates) that

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$$

iii. Consider some arbitrary material control volume $V(t)$ whose surface is $S(t)$; t is time. Use Cartesian tensor methods to show that

$$\int_{S(t)} \mathbf{r} \times (\mathbf{n} \cdot \mathbf{X}) dS = \int_{V(t)} (\mathbf{r} \times (\nabla \cdot \mathbf{X}) - \epsilon{:}\mathbf{X}) dV$$

where $\mathbf{X}$ is a second order tensor, $\mathbf{n}$ is the outward normal to the surface, $\mathbf{r}$ is the position vector of a material point in $V(t)$ and the components of ϵ are ε_{ijk}.

iv. Let the vector

$$\mathbf{w} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$$

where $\mathbf{v}$ is arbitrary and $\mathbf{n}$ is a unit vector. In what direction does $\mathbf{w}$ point and how is it related to $\mathbf{v}$?

Question 4

Starting from the general solution of Laplace's equation $\nabla^2 \phi = 0$ in spherical coordinates assuming symmetry about the axis $\theta = 0$:

$$\phi = \sum_{n=0}^{\infty} [A_n r^{-(n+1)} + B_n r^n] P_n(\cos\theta)$$

find the irrotational incompressible flow of an inviscid fluid about a sphere of radius a in a uniform stream of magnitude U. Find the slip (tangential) velocity on the surface of the sphere, the pressure along the surface of the sphere and compute the drag on the sphere explicitly (by integrating the stress tensor).

Hint: write down the boundary conditions on the velocity and hence for ϕ and use them to find the coefficients A_n, B_n.