Department of Mathematics and Statistics
620–342
Industrial and Applied Mathematics

Assignment 2. Due: October 31

This assignment counts for 15% of the marks for this subject.

Question 1

Use the method of separation of variables to solve the problem of start-up flow in an infinite circular pipe of radius a. This means that the fluid is at rest in the pipe until $t = 0$ when a constant pressure gradient G is imposed. You may start from the governing equation for unidirectional flow

$$\rho \frac{\partial w}{\partial t} = G(t) + \mu \nabla_2^2 w = G(t) + \mu \frac{1}{\sigma} \frac{\partial}{\partial \sigma} (\sigma \frac{\partial w}{\partial \sigma})$$

i. Write the solution as a steady-state solution $w^{ss}(\sigma)$ plus a transient $w^{tr}(\sigma, t)$. Find $w^{ss}(\sigma)$ and write down the equations for $w^{tr}(\sigma, t)$.

ii. Nondimensionalize the equations using $l_c = a$, $u_c = \frac{Ga^2}{\mu}$ with t_c left undecided. What choice of t_c simplifies the equations for $w^{tr}(\sigma, t)$?

iii. With this choice of t_c, solve for $w^{tr}(\sigma, t)$ by separation of variables to show

$$w(\sigma, t) = \frac{Ga^2}{4\mu}[(1 - (\frac{\sigma}{a})^2) - \sum_{n=1}^{\infty} \frac{8}{x_n^3} \frac{1}{J_1(x_n)} \exp(-x_n^2 \nu t/a^2) J_0(x_n \frac{\sigma}{a})]$$

where x_n is the n^{th} positive real root of the Bessel function J_0. How long does it take the flow to get close (say within 1 %) to the steady flow along the centreline? (Assume the first term dominates and look up x_1 in tables.)

You will need to find some integrals of Bessel functions. I suggest you look for orthogonality relations and recurrence relations for Bessel functions in, say, Abramowitz & Stegun Eqs. 11.4.5, 9.1.27,11.3.20

Question 2

Since the stress is infinite at $t = 0$ in the Rayleigh problem, consider a similar problem except that at $t = 0$ the plate is suddenly subjected to a constant shear stress τ from then on. Is there a natural velocity scale for this problem?

Hence argue why you might look for a similarity solution of the form

$$w = \frac{\tau}{\mu} y F(\xi)$$

or

$$w = \frac{\tau}{\mu} \sqrt{\nu t} F(\xi)$$

where

$$\xi = \frac{y}{\sqrt{\nu t}}$$

From the first of these forms, solve for $F(\xi)$ and hence show that

$$w = \frac{\tau}{\mu}\sqrt{\nu t}[\frac{2}{\sqrt{\pi}}\exp(-\xi^2/4) - \xi\text{erfc}(\xi/2)]$$

where $\text{erfc}(x) = 1 - \text{erf}(x)$.
Hint: in applying the boundary conditions it may be easier to use the second form.
What problem would you face if you tried to program a plate to move in this way i.e. so as to produce a constant stress?

Question 3

Use matched asymptotic expansions to solve to $O(\epsilon)$

$$\epsilon y'' + y' = -\exp(-x), y(0) = 0, y(1) = 1$$

by the following procedure, assuming there is a boundary layer at $x = 0$:

i. Calculate two terms of the outer solution i.e. $y^{out} = y_0 + \epsilon y_1 + O(\epsilon^2)$, and two terms of the inner solution. (you will have to determine a suitable scaling for the inner solution using the method of dominant balance)

ii. Match in the following way: rewrite y^{out} in the inner variable X, expand for small ϵ and keep only two terms - this gives $(y^{out})^{in}$ to $O(\epsilon^2)$.
Similarly for y^{in} - rewrite in the outer variable x, expand for small ϵ to $O(\epsilon^2)$ - this gives $(y^{in})^{out}$ to $O(\epsilon^2)$.
(you can use the fact that terms like $e^{-x/\epsilon}$ are smaller than any power of ϵ as $\epsilon \to 0$.)
Matching these determines all the constants in y^{in}.

iii. Hence construct a composite solution

$$y^c = y^{in} + y^{out} - (y^{in})^{out}$$

Compare graphically/numerically the composite solution to O(1):

$$y^c = 1 + e^{-x} - e^{-1} - e^{-x/\epsilon}(2 - e^{-1})$$

and your $O(\epsilon)$ solution with the exact solution (hint: first year) for some suitable ϵ, say $\epsilon = 0.05$.