Departmen t of Mathematics and Statistics
620–342
Industrial and Applied Mathematics

2008

Problem Sheet 0. Revision of Vector Analysis

1 Problems taken from 620-231

1. Flow Lines. Sketch a few flow lines of the following vector fields. Derive the differential equations for the flow lines and solve them to obtain an expression for the flow lines.

(a) $\mathbf{f}(x,y) = (y,-x)$; (b) $\mathbf{f}(x,y) = (x,-y)$; (c) $\mathbf{f}(x,y,z) = (y,-x,0)$.

2. Divergence. Find the divergence of the following vector fields:

(a) $\mathbf{V}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$;
(b) $\mathbf{V}(x,y,z) = x\mathbf{i} + (y+\cos x)\mathbf{j} + (z+e^{xy})\mathbf{k}$;
(c) $\mathbf{V}(x,y) = \sin(xy)\mathbf{i} - \cos(x^2y)\mathbf{j}$.

3. Curl. Find the curl of the following vector fields:

(a) $\mathbf{V}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;
(b) $\mathbf{V}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$;
(b) $\mathbf{V}(x,y,z) = (x^2+y^2+z^2)(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$;

4. Div, Grad and Curl. Let $\mathbf{F} = 2xz^2\mathbf{i} + \mathbf{j} + y^3zx\mathbf{k}$, $f = x^2y$. Compute:

(a) ∇f; (b) $\nabla \times \mathbf{F}$; (c) $\mathbf{F} \times \nabla f$; (d) $\mathbf{F} \cdot \nabla f$.

5. Curl of Grad. Show that $\nabla \times (\nabla f) = \mathbf{0}$ for the following functions:

(a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$; (b) $f(x,y,z) = xy + yz + xz$.

6. Divergence Properties. Suppose that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$. Which of the following must have zero divergence?

(a) $\mathbf{F} + \mathbf{G}$; (b) $\mathbf{F} \times \mathbf{G}$.

7. Vector Properties. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Prove using the standard identities of vector analysis that

(a) $\nabla^2 \left(\frac{1}{r}\right) = 0, \quad r \neq 0;$ (b) $\nabla^2 (r^n) = n(n+1)r^{n-2};$ (c) $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 0;$

(d) $\nabla \cdot (r^n \mathbf{r}) = (n+3)r^n;$ (e) $\nabla \times \left(\frac{\mathbf{r}}{r}\right) = \mathbf{0};$ (f) $\nabla \times (r^n \mathbf{r}) = \mathbf{0}.$

where n is any real number.

8. Line Integral Properties. Let $\mathbf{c}$ be a smooth differentiable path.

(a) If $\mathbf{F}$ is perpendicular to $\mathbf{c}'(t)$ at $\mathbf{c}(t)$, show that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0.$

(b) If $\mathbf{F}$ is parallel to $\mathbf{c}'(t)$ at $\mathbf{c}(t)$, show that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} |\mathbf{F}| ds.$

9. Conservative Fields. In each case, show that $\mathbf{F}$ is a conservative vector field and find a scalar function ϕ such that $\mathbf{F} = \nabla\phi$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ along paths joining $(1, -2, 1)$ to $(3, 1, 4)$.

(a) $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k};$
(b) $\mathbf{F}(x, y, z) = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}.$

10. Cylindrical Coordinates. Define curvilinear coordinates (ρ, ϕ, z) by

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z.$$

where $\rho \geq 0, 0 \leq \phi \leq 2\pi$. If n is an integer, evaluate the following quantities:

(a) $\nabla\phi$ (b) $\nabla\rho^n$ (c) $\nabla^2(\rho^2\cos\phi)$.

Note that in 620342, we denote cylindrical coordinates by (σ, ϕ, z) since we reserve ρ for the density.

11. Spherical Coordinates. Define curvilinear coordinates (r, θ, ϕ) by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

where $r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$. Evaluate the following quantities:

(a) $\nabla\phi$ (b) $\nabla\theta$ (c) $\nabla \cdot (\hat{\mathbf{r}}\cot\phi - 2\hat{\phi})$.

12. Coordinate Systems. Using spherical coordinates, express each of the orthonormal vectors $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\phi}$ in terms of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ and (x, y, z).

2 Other revision/background problems

Question 1

Show

i. $\nabla \times \nabla\phi = 0$

ii. $\nabla \cdot \nabla \times \mathbf{a} = 0$

iii. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

iv. $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$

where $\mathbf{u} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ and $\mathbf{u} = (u, v, w)$.

Question 2

Find $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$ where

i. $\mathbf{u} = (x, -y)$

ii. $\mathbf{u} = \frac{1}{\sqrt{x^2+y^2}}(x, -y)$

Question 3

Are the following vector fields the gradients of scalar fields? If so, find the scalar fields.

i. $\mathbf{u} = (x, -y)$

ii. $\mathbf{u} = (y, -x)$

iii. $\mathbf{u} = (0, 0, xyz)$

iv. $\mathbf{u} = (y, x)$

v. $\mathbf{u} = \frac{1}{\sqrt{x^2+y^2}}(x, y)$

Question 4

The inverse square law vector field $\mathbf{F}$ has the general form

$$\mathbf{F} = a\frac{\mathbf{r}}{\mid \mathbf{r} \mid^3}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and a is a constant.

Prove that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$ except at $\mid \mathbf{r} \mid = \mathbf{0}$. Show that $\mathbf{F}$ is a conservative field by finding a potential U such that $\mathbf{F} = -\nabla \mathbf{U}$ and verify that $\nabla^2 U = 0$.

Question 5

Define the Laplacian of a *vector field* (or **vector Laplacian**) by

$$\nabla^2(\phi\mathbf{a}) = (\nabla^2\phi)\mathbf{a}$$

where $\mathbf{a}$ is a constant vector. Hence, in Cartesian coordinates,

$$\nabla^2\mathbf{u} = \nabla^2(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = (\nabla^2 u)\mathbf{i} + (\nabla^2 v)\mathbf{j} + (\nabla^2 w)\mathbf{k}.$$

Show that

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}$$

This identity is used to define $\nabla^2\mathbf{u}$ in other coordinates. Use it to find $\nabla^2\mathbf{u}$ in cylindrical coordinates.

Question 6

From the divergence theorem, prove that $\int \phi \, d\mathbf{S} = \int \nabla\phi \, dV$.

Question 7

Consider the vector field

$$\mathbf{u} = \frac{1}{x^2+y^2}(-y, x)$$

defined for $x^2 + y^2 > 0$.

Show that $\nabla \times \mathbf{u} = \mathbf{0}$ for $(x, y) \neq (0, 0)$ but that the line integral of $\mathbf{u}$ around the circle $x^2 + y^2 = a^2, z = 0$ is non-zero. Is there a contradiction here with the results of Question 4?

Question 8

Consider the vector $\mathbf{q} = q(\theta)\mathbf{k}$ where $\mathbf{k}$ is the unit vector in the z direction and θ is the spherical polar angle,

i. express the integral

$$\int_S d\mathbf{S} \cdot \mathbf{q}$$

in terms of the r and θ components of $\mathbf{q}$ if S is the surface of a sphere of radius a

ii. hence evaluate the integral if $q(\theta) = \cos\theta$

Question 9

The torque (about the origin) acting on a particle is defined by

$$\Lambda = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ is the position vector of the point of application of the force $\mathbf{F}$, relative to the origin. The total torque on a rigid body is found by integrating the torque acting on each element of the body. In fluid mechanics, the torques ususlly act only on the surface of the body.

i. Suppose a cylinder of radius a is rotating about its axis so that the drag force produces a force acting on each surface element

$$\mathbf{F} = -D\mathbf{e}_\phi$$

Find the total torque acting on the cylinder, per unit length.

ii. Suppose a sphere of radius a is rotating about the z-axis so that the drag force produces a force acting on each surface element

$$\mathbf{F} = -D\mathbf{e}_\phi$$

Find the total torque acting on the sphere.