Departmen t of Mathematics and Statistics
620–342
Industrial and Applied Mathematics

2008

Problem Sheet 0. Some answers

1 Problems taken from 620-231

1. Flow Lines.
(a) $x^2 + y^2 = r^2$ where r is a constant
(b) $y = \frac{c}{x}$ where c is a constant
(c) level curves are concentric circles $x^2 + y^2 = r^2$ where r is a constant

2. Divergence.
(a) 0 (b) 3 (c) $y \cos(xy) + x^2 \sin(x^2 y)$

3. Curl.
(a) $\mathbf{0}$ (b) $\mathbf{0}$ (c) $(10y - 8z)\mathbf{i} - (10x - 6z)\mathbf{j} + (8x - 6y)\mathbf{k}$

4. Div, Grad and Curl.
(a) $2xy\mathbf{i} + x^2\mathbf{j}$
(b) $3xy^2z\mathbf{i} + (4xz - y^3z)\mathbf{j}$
(c) $-y^3zx^3\mathbf{i} + 2y^4x^2z\mathbf{j} + (2x^3z^2 - 2xy)\mathbf{k}$
(d) $4x^2yz^2 + x^2$

5. Curl of Grad.

6. Divergence Properties.
(a) yes (b) no

7. Vector Properties.

8. Line Integral Properties.

9. Conservative Fields.
(a) $f = x^2yz - \cos x + c$, $38 - \cos 3 + \cos 1$ (b) $f = xz^3 + x^2y + c$, 202

10. Cylindrical Coordinates.
(a) $\frac{1}{\rho}(-\sin\phi, \cos\phi, 0)$ (b) $n\rho^{n-1}(\cos\phi, \sin\phi, 0)$ (c) $3\cos\phi$

11. Spherical Coordinates.
(a) $\frac{1}{r\sin\theta}\hat{\phi}$ (b) $\frac{1}{r}\hat{\theta}$ (c) $\frac{2\cot\phi}{r}$

12. Coordinate Systems.

$$\hat{\mathbf{r}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}, \quad \hat{\phi} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}, \quad \hat{\theta} = \frac{xz\mathbf{i} + yz\mathbf{j} - (x^2+y^2)\mathbf{k}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$$

2 Other revision/background problems

Question 1

Straightforward but tedious. Use Cartesian coordinates.

Question 2

i. $\nabla \cdot \mathbf{u} = 0$ and $\nabla \times \mathbf{u} = 0$

ii. $\nabla \cdot \mathbf{u} = \frac{y^2 - x^2}{r^3}$ and $\nabla \times \mathbf{u} = \frac{2xy}{r^3}\mathbf{k}$

Moral: multiplying a vector field by a scalar field can have a drastic effect on $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$

Question 3

i. Yes

ii. No

iii. No

iv. Yes

v. Yes

Question 4

$U = \frac{a}{r}$

Question 5

Straightforward but tedious. Use Cartesian coordinates.

For $\nabla^2\mathbf{u}$ in cylindrical coordinates, see e.g. Appendix 2 of Batchelor or the formula sheet.

Question 6

Choose $\mathbf{u} = \phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector, in the usual divergence theorem.

Question 7

There is no contradiction here with the results of Question 4. Hint: think “simply connected”

Question 8

$\frac{4\pi}{3}a^2$

Question 9

i. $-2\pi a^2 D\mathbf{k}$

ii. $-\pi^2 a^3 D\mathbf{k}$