Department of Mathematics and Statistics
620–342
Industrial and Applied Mathematics

Problem Sheet 2.
Cartesian tensors

Question 1

Which of the following expressions make sense in index notation?

i. $a = b_i c_{ij} d_j$

ii. $a_i = b_i + c_{ij} d_{ji} e_i$

iii. $a = b_i c_i + d_j$

iv. $a_l = \varepsilon_{ijk} b_j c_k$

v. $a_k = b_i c_{ki}$

vi. $a_{ij} = b_i c_j + e_{jk}$

Question 2

Simplify the expression

$$(A_{ijk} + A_{jki} + A_{kji}) x_i x_j x_k$$

Question 3

Use Cartesian tensor methods to prove the following identities:

i. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

ii. $\nabla \times \nabla \phi = 0$

iii. $\nabla \cdot \nabla \times \mathbf{a} = 0$

iv. $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{u}$

v. $\nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho$

vi. $\nabla \times \mathbf{r} = 0$ where $\mathbf{r} = (x, y, z)$

vii. $\mathbf{a} \times \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} \times \mathbf{b} = 0$

viii. $\frac{D}{Dt}(\mathbf{r} \times \rho \mathbf{u}) + (\mathbf{r} \times \rho \mathbf{u}) \nabla \cdot \mathbf{u} = \mathbf{r} \times [\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u})]$

ix. $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u}$

x. $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$

Hint: use the identity

$$\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$$

freely. Convert the identity to component form, manipulate then convert back to dyadic form.

Question 4

Prove or disprove , where b_{ij} is not symmetric:

i. $b_{ij}x_iy_j = b_{ij}y_ix_j$

ii. $b_{ij}x_ix_j = b_{ji}x_ix_j$

iii. $(b_{ij} + b_{ji})x_iy_j = 2b_{ji}x_iy_j$

iv. $(b_{ij} + b_{ji})x_ix_j = 2b_{ji}x_ix_j$

v. $\varepsilon_{ijk}\tau_{ij} = 0 \Rightarrow \tau_{ij} = \tau_{ji}$

Question 5

Show that

i. $\mathsf{S}{:}\mathsf{T} = 0$ if S is symmetric and T is antisymmetric.

ii. $\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl}$

Question 6

Evaluate

i. δ_{ii}

ii. $\delta_{ij}\delta_{ji}$

iii. $\delta_{ij}\delta_{ik}\delta_{jk}$

iv. $\varepsilon_{ijk}\varepsilon_{ijk}$

Question 7

A particle is rotating with constant angular velocity $\mathbf{\Omega}$ about a point P so that its velocity is $\mathbf{u} = \mathbf{\Omega} \times \mathbf{r}$. Use Cartesian tensor methods to show that its acceleration can be written as either

$$\mathbf{a} = (\mathbf{\Omega} \cdot \mathbf{r})\mathbf{\Omega} - (\mathbf{\Omega} \cdot \mathbf{\Omega})\mathbf{r}$$

or

$$\mathbf{a} = -\frac{1}{2}\nabla[(\mathbf{\Omega} \times \mathbf{r})^2]$$