Department of Mathematics and Statistics
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Industrial and Applied Mathematics

Problem Sheet 3.
Hydrostatics/Bernoulli equation/potential flow

Hydrostatics

Question 1

Consider a solid body partially immersed at the horizontal surface between two fluids of density ρ_1 and ρ_2. Derive the condition that the body experiences no net force, assuming constant gravity. Assume the pressure is continuous at the interface.

Question 2

Atmospheric pressure is about 1 bar = $10^5 Nm^{-2}$. How deep underwater do you have to dive before you feel a pressure of 2 bar? What is the pressure at the bottom of a backyard swimming pool?

Question 3 Isothermal atmosphere

The atmosphere can be well approximated by a perfect gas, for which $p = \rho RT$, where R is the gas constant. Assume that the atmosphere is isothermal ($T = T_s$). Show that, under hydrostatic equilibrium, the pressure is given by

$$p = p_s exp(-z/H_s)$$

where $H_s = RT_s/g$ is called the scale height of the atmosphere. Estimate the length scale H_s.
The lower troposphere (10–30 km above sea level) is roughly isothermal. The stratosphere (0–10 km above sea level) is more nearly adiabatic (no heat flow) than isothermal.

Bernoulli equation

Question 4

An inviscid fluid is rotating under gravity with constant angular velocity Ω so that relative to fixed Cartesian axes $\mathbf{u} = -\Omega y\mathbf{i} + \Omega x\mathbf{j}$. We wish to find the surfaces of constant pressure and hence the surface of a uniformly rotating bucket of water (which will be at atmospheric pressure).

'By Bernoulli', $\frac{1}{2}q^2 + \frac{p}{\rho} + gz$ is constant, so the constant pressure surfaces are

$$z = C - \frac{\Omega^2}{2g}(x^2 + y^2)$$

But this means the surface of a rotating bucket of water has its highest point in the middle! What is wrong?

Write down the Euler equations in component form, integrate them directly to find the pressure p and hence obtain the correct shape of the free surface.

Question 5

Show that for unsteady *irrotational* flow we have a Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + \frac{p}{\rho} + \chi = F(t)$$

where ϕ is the velocity potential and $\mathbf{b} = -\nabla\chi$.

Show that the choice of the function $F(t)$ does not affect the velocity field.

Question 6

A compressible fluid is called *barotropic* if the pressure is a function of density alone viz. $p = f(\rho)$. Derive a Bernoulli equation along a streamline for the case of steady flow of a barotropic fluid.

A special case of this is *homentropic flow* for which $p = k\rho^\gamma$ where $\gamma \approx 1.4$. Derive a Bernoulli equation for this case and explain why the pressure term is more important in this case than for an incompressible fluid.

Potential flow

Question 7

Show that the Stokes streamfunction of an axisymmetric potential flow satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = 0$$

which is *n*ot Laplace's equation.

Question 8

i. Investigate the potential flow given by the velocity potential in spherical coordinates

$$\phi = Ur\cos\theta - \frac{m}{r}$$

where U, m are constants.

ii. Show that the same flow can be described by the Stokes streamfunction

$$\Psi = -\frac{1}{2}Ur^2\sin^2\theta + m\cos\theta$$

iii. Find the pressure field for this velocity field.