Department of Mathematics and Statistics
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Industrial and Applied Mathematics

Problem Sheet 4.
Vorticity and rate of strain

Question 1

A swirling flow along a pipe is assumed to have velocity

$$\mathbf{u} = U(\sigma)\mathbf{e}_\phi + V(\sigma)\mathbf{e}_z$$

in cylindrical polars. Does this satisfy the vorticity equation and the continuity equation?

Question 2

Compute the vorticity and rate of strain tensor in cylindrical coordinates for the flows

i. $\mathbf{u} = (1-\sigma^2)\mathbf{k}$

ii. $\mathbf{u} = \frac{1}{\sigma}\mathbf{e}_\phi$

iii. $\mathbf{u} = \frac{1}{\sigma}(1-\exp(-\sigma^2/t))\mathbf{e}_\phi$

Question 3

The (2D) streamfunction

$$\psi = U\sigma\sin\phi - Ua^2\sin\phi/\sigma$$

gives a model flow past a cylinder $\sigma = a$. Find the components of the rate-of-strain tensor (in cylindrical polar coordinates) — use the formula sheet — and hence find the components of the deviatoric stress tensor. Hence compute the viscous force acting on the cylinder. Is it realistic?

Question 4

The streamfunction in plane polar coordinates

$$\psi = U\sigma\sin\phi - Ua^2\sin\phi/\sigma - \frac{1}{2}\tau\sigma^2\sin^2\phi - \frac{1}{4}\tau a^4\cos 2\phi/\sigma^2$$

represents a flow round the cylinder $\sigma = a$ (where $\psi = -\frac{1}{4}\tau a^2$). Show that, at large distances upstream the vorticity is uniform and verify that the vorticity is everywhere constant. Calculate the force due to the pressure on the cylinder.

Question 5

i. Show that the Newtonian constitutive equation is equivalent to

$$\mathbf{t} = -p\mathbf{n} + \mu[2(\mathbf{n}\cdot\nabla)\mathbf{u} + \mathbf{n}\times(\nabla\times\mathbf{u})]$$

ii. Verify that for a simple shear flow $\mathbf{u} = (u(y), 0, 0)$ we recover

$$\mathbf{t} = (\mu \frac{du}{dy}, -p, 0)$$

if $\mathbf{n} = (0, 1, 0)$

iii. Show that for pure rotary flow $\mathbf{u} = u_\phi(\sigma)\mathbf{e}_\phi$ we recover

$$\mathbf{t} = -p\mathbf{e}_\sigma + \mu\sigma \frac{d}{d\sigma}(\frac{u_\phi}{\sigma})\mathbf{e}_\phi$$

and show that the second term vanishes for uniform rotation.

iv. Show explicitly that the terms $2\mu e_{ij}$ of the stress tensor vanish for the case of uniform rotation $\mathbf{u} = \mathbf{\Omega} \times \mathbf{r}$

Question 6

Consider the flow

$$\mathbf{u} = (1 - y^2)\mathbf{i}$$

Is the flow incompressible? irrotational?

Find the rate of strain tensor and hence the principal rates of strain and principal axes (eigenvalues and eigenvectors of the rate of strain tensor). Hence sketch the local deformation undergone by a spherical blob of fluid.

Question 7

i. Show that, in cylindrical coordinates,

$$\frac{\partial}{\partial \phi}\mathbf{e}_\sigma = \mathbf{e}_\phi$$

$$\frac{\partial}{\partial \phi}\mathbf{e}_\phi = -\mathbf{e}_\sigma$$

but all other derivatives of the basis vectors vanish.

ii. Hence by applying ∇ directly, find $\nabla\mathbf{u}$ and e in cylindrical coordinates.

iii. By using the identity

$$\frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u})$$

show that, if $\mathbf{u} = u\mathbf{e}_\sigma + v\mathbf{e}_\phi + w\mathbf{e}_z$, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ takes the form:

$$(u\frac{\partial u}{\partial \sigma} + \frac{1}{\sigma}v\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{v^2}{\sigma})\mathbf{e}_\sigma + (u\frac{\partial v}{\partial \sigma} + \frac{1}{\sigma}v\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{uv}{\sigma})\mathbf{e}_\phi + (u\frac{\partial w}{\partial \sigma} + \frac{1}{\sigma}v\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z})\mathbf{e}_z$$

iv. Evaluate $\nabla^2\mathbf{u}$ in cylindrical coordinates by using the identity

$$\nabla \times (\nabla \times \mathbf{q}) = \nabla(\nabla \cdot \mathbf{q}) - \nabla^2\mathbf{q}$$

v. Hence write down the Navier-Stokes equations in cylindrical coordinates (see Appendix C of Notes)

vi. *(If keen) Repeat for spherical polar coordinates.