Departmen t of Mathematics and Statistics
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Problem Sheet 4. Some answers

Question 1

Just evaluate both sides of the vorticity equation, remembering that

$$\frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\sigma$$

Question 2

Use formula sheet.

i. $\boldsymbol{\omega} = 2\sigma \mathbf{e}_\varphi$; $e_{\sigma z} = e_{z\sigma} = -\sigma$, the other components vanish.
This is Poiseuille flow.

ii. $\boldsymbol{\omega} = 0$; $e_{\sigma\varphi} = e_{\varphi\sigma} = -\frac{1}{\sigma^2}$, the other components vanish.
This is a line vortex or free vortex.

iii. $\boldsymbol{\omega} = \frac{2}{t} \exp^{-\sigma^2/t} \mathbf{e}_\varphi$; $e_{\sigma z} = e_{z\sigma} = \frac{\sigma}{t} \exp^{-\sigma^2/t} - \frac{1}{\sigma^2}(1 - \exp^{-\sigma^2/t})$, the other components vanish.
This is a diffusing line vortex.

Question 3

$e_{\sigma\sigma} = \frac{2Ua^2}{\sigma^3} \cos\varphi$; $e_{\sigma\varphi} = \frac{2Ua^2}{\sigma^3} \sin\varphi$; $e_{\varphi\varphi} = -\frac{2Ua^2}{\sigma^3} \cos\varphi$

The viscous force per unit length is given by

$$F = 2\mu \int_0^{2\pi} d\varphi \; a[\cos\varphi e_{\sigma\sigma} - \sin\varphi e_{\sigma\varphi}]$$

The force is zero. The flow is not realistic (it's actually potential flow past a cylinder).

Question 4

Express the vorticity z-component in terms of the streamfunction: you should get $\omega_z = \tau$ everywhere. This agrees with the previous result that the vorticity is constant along streamlines for steady 2D flow. Then the fact that the vorticity is constant far upstream (i.e. has the same value on different streamlines) implies the vorticity is constant throughout.

Integrate the stress tensor (viz. the pressure) over the cylinder. You only need the pressure on the surface so use Bernoulli on the streamline that goes round the cylinder ($\psi = -\frac{1}{4}\tau a^2$).

You should find that the drag is zero (agreeing with D'Alembert's paradox) but the lift (per unit span) is $2\pi a^2 \rho U \tau$.

Question 5

Question 6

Incompressible but not irrotational.

$\boldsymbol{\omega} = 2y\mathbf{k}$; $e_{xy} = e_{yx} = -y$

so principal rates of strain are $0, +y, -y$ with principal axes **k**,**i**+**j**,**i**-**j**, respectively.

Question 7