Department of Mathematics and Statistics
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Industrial and Applied Mathematics

Problem Sheet 5.
Navier-Stokes equations I

Question 1

Consider the flow

$$\mathbf{u} = \frac{\Omega a^2}{\sigma}\mathbf{e}_\phi, \sigma \geq a$$

outside a rotating cylinder. What is wrong with the following argument?

'The flow is irrotational so the viscous forces vanish so there is no torque on the cylinder'.

Question 2

Show that the normal viscous stress at a stationary solid boundary must vanish for an incompressible Newtonian fluid. You may assume the boundary is flat but this is unnecessary.

Hint: use the continuity equation.

Question 3

Estimate the order of magnitude of the Reynolds number for

- flow past a jumbo jet (150 ms^{-1})
- a thick layer of golden syrup oozing off a spoon
- sperm (tail length 10 μm) swimming at $10^{-4} ms^{-1}$ in water
- a person walking
- an ant walking

Question 4

Solve the startup Couette problem i.e a lower boundary $y = 0$ is suddenly moved with speed u while an upper boundary $y = d$ is held stationary. To make the boundary conditions homogeneous, write the solution as a steady-state solution (steady Couette flow) and a transient solution. Solve for the transient solution by separation of variables. Show that the stress at the lower boundary decreases monotonically from its intial (infinite) value to the steady state $\frac{\mu U}{d}$.

Question 5

For unidirectional flow down an annular pipe with inner surface $\sigma = a$ and outer surface $\sigma = 2a$, find the steady flow profile $w(\sigma)$ caused by an axial pressure gradient $\frac{dp}{dz} = -G$ and determine the stress vector acting at both inner and outer surfaces. Also find the volume flow rate through the pipe.

Question 6

Suppose viscous fluid occupies the region $\sigma \leq a$ within a circular cylinder of radius a and suppose that the cylinder and fluid are rotating with uniform angular velocity Ω so that

$$u_\phi = \Omega\sigma, \sigma \leq a, t = 0$$

The cylinder is suddenly brought to rest. Use separation of variables to show that the motion is given by

$$u_\phi(\sigma, t) = -2\Omega a \sum_{n=1}^{\infty} \frac{J_1(\lambda_n \sigma/a)}{\lambda_n J_0(\lambda_n)} \exp(-\lambda_n^2 \nu t/a^2)$$

where λ_n is the n^{th} positive real root of $J_1(x)$. Given that the first term gives the dominant rate of decay, estimate the 'spin-down' time of a cup of tea/coffee and compare with your observations. Interpret in terms of vorticity difusion.

Hint: you only need to consider the azimuthal component of the Navier-Stokes equation (see Appendix C).