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Industrial and Applied Mathematics

Problem Sheet 6.
Navier-Stokes equations II

Question 1

Two incompressible, viscous and immiscible fluids of the same density ρ flow, one on top of the other, down an inclined plane making an angle α with the horizontal. Their viscosities are μ_1 and μ_2, the lower fluid is of depth h_1 and the upper fluid is of depth h_2. What are the boundary conditions at each fluid-fluid interface?

Show that

$$u_1(y) = [(h_1 + h_2)y - \frac{1}{2}y^2]\frac{g \sin\alpha}{\nu_1}$$

so that the velocity of the lower fluid $u_1(y)$ is dependent on the depth h_2 but not the viscosity of the upper fluid. Why is this?

Question 2

A cylindrical rod of radius a is being pulled upward with velocity U after it has been coated with a liquid film of thickness h. The coated liquid is draining downwards due to gravity. Show that the velocity profile across the film is

$$u_z = U + \frac{g}{4\nu}(\sigma^2 - a^2 - 2(a+h)^2 \ln(\sigma/a))$$

Question 3

Viscous fluid occupies the region $0 < z < h$ between two rigid walls $z = 0$ and $z = h$. The lower boundary is at rest , the upper boundary rotates with constant angular velocity ω about the z-axis. Show that a steady solution of the Navier-Stokes equations of the form $\mathbf{u} = u_\phi(\sigma, z)\mathbf{e}_\phi$ is not possible. i.e. any rotating motion $u_\phi(\sigma, z)$ must be accompanied by a secondary flow ($u_\sigma, u_z \neq 0$).

You need to start from the Navier-Stokes equations in cylindrical cordinates.

Question 4

i. A liquid is in the annular space between two vertical cylinders of radii $\kappa R, R$, and the liquid is open to the atmosphere at the top. Show that when the inner cylinder rotates with angular velocity Ω_i and the outer cylinder is fixed, the free liquid surface has the shape

$$z_R - z = \frac{1}{2g}(\frac{\kappa^2 R\Omega_i}{1-\kappa^2})^2(\xi^{-2} + 4\ln\xi - \xi^2)$$

where z_R is the height of the liquid at the outer cylinder wall and $\xi = \sigma/R$.

ii. Repeat part i but with the inner cylinder fixed and the outer cylinder rotating with angular velocity Ω_o. Show that the shape of the free liquid surface is

$$z_R - z = \frac{1}{2g}(\frac{\kappa^2 R\Omega_0}{1-\kappa^2})^2[(\xi^{-2} - 1) + 4\kappa^{-2}\ln\xi - \kappa^{-4}(\xi^2 - 1)]$$

Question 5

Find the velocity field (there is only an azimuthal component) for fluid confined between two concentric *spheres* of radius $\kappa R, R$ where $\kappa < 1$, rotating with angular velocities Ω_i, Ω_o respectively.

Question 6

In classical microelectrophoresis, measurements are done in a cell consisting of two parallel planes a distance $2h$ apart i.e at $z = \pm h$. The flow in the cell consists of

i. a uniform flow of strength U_{eo} (electroosmotic flow)

ii. a plane Poiseuille flow opposing the electroosmotic flow, of sufficient strength to produce zero nett volume flux (because the cell is actually closed)

Find the combined flow field and sketch it. Show that there are two 'stationary layers' (where $v = 0$) at $z = \pm h/\sqrt{3}$.