

Department of Mathematics and Statistics
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Industrial and Applied Mathematics

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Problem Sheet 6. Some answers

Question 1

This problem is from Acheson.

At the free surface, we have a zero shear stress condition. At the liquid/liquid boundary, we have continuity of velocity and shear stress. This means that

$$u_1(h_1) = u_2(h_1)$$

and

$$\mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy} \Big|_{h_1}$$

where y is the coordinate normal to the inclined plane.

The answer is the same as if you ignored the conditions at $y = h_1$ and just solved for a single fluid of depth $h_1 + h_2$. Why?

Recall that for steady unidirectional flow, the fluid has no acceleration so the forces acting on the fluid must balance. In the top fluid, there is no force from the free surface (zero stress condition) so the shear stress from the bottom fluid is balanced by the gravity force on the top fluid.

In the bottom fluid, the shear stress from the top fluid is equal to the gravity force on the top fluid which is proportional to its mass, hence h_2 .

Question 2

Start from Eq. 7.2 in cylindrical coordinates but include gravity (it's a gravity-driven flow). You can neglect the pressure gradient due to the pressure continuity condition at the free surface (as in §7.4).

Then use no-slip at the rod and zero stress BC's at the free surface.

Question 3

Look at the Navier-Stokes equations in cylindrical coordinates assuming the existence of a flow

$$\mathbf{u} = u_\phi(\sigma, z)\mathbf{e}_\phi$$

The z -component says that the dynamic pressure has no z -dependence (the hydrostatic pressure does, of course). Then the σ -component says that the velocity cannot depend on z , which contradicts the boundary conditions.

Question 4

A straightforward application of the results in Section 7.8 of the Notes. The general form of the velocity field and pressure field is given there (p. 101).

Just specialize the coefficients for the special cases of outer/inner cylinder being fixed and use the fact that a free surface is a surface of constant pressure.

Question 5

Look at the Navier-Stokes equations in spherical coordinates assuming the existence of a flow

$$\mathbf{u} = w(r, \theta) \mathbf{e}_\phi$$

You should get after some work

$$\nabla^2 w = \frac{w}{r^2 \sin^2 \theta}$$

Note: a scalar Laplacian.

Both boundary conditions at inner and outer surfaces take the form

$$w = K \sin \theta$$

which suggests we look for a flow of the form

$$w = f(r) \sin \theta$$

Substituting this form produces an equidimensional equation which gives the general solution

$$f = c_1 r + c_2 r^{-2}$$

The answer follows from applying the boundary conditions.

Question 6

$$v = U_{eo} \left[1 - \frac{3}{2} \left(1 - \left(\frac{z}{h} \right)^2 \right) \right]$$