Department of Mathematics and Statistics
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Industrial and Applied Mathematics

Problem Sheet 7.
Creeping flow

Question 1

Use separation of variables to find the general solution to $E^2\Psi = 0$ where E^2 is the axisymmetric Stokes operator in spherical coordinates.

Hint: for the angular part, compare the differential equation

$$(1-x^2)\mathcal{Q}'' + n(n+1)\mathcal{Q} = 0$$

with the corresponding equation for the axisymmetric Laplacian in spherical coordinates (see 331 notes) in the variable $x = \cos\theta$.

Question 2

(a) Show that, for a sphere of radius a rotating with angular velocity $\Omega\hat{\mathbf{k}}$, the points on the surface have velocity $\mathbf{U} = \Omega a \sin\theta \mathbf{e}_\phi$, in spherical polar coordinates.

(b) 'Guided by the form of the boundary condition', consider a velocity field of the form

$$\mathbf{u} = f(r)\sin\theta\mathbf{e}_\phi$$

Show that such a flow satisfies the continuity equation for incompressible flow and express $\nabla \times \nabla \times \boldsymbol{\omega}$ in terms of f (and its derivatives).

(c) Hence show that the velocity field outside a sphere of radius a rotating with angular velocity $\Omega\hat{\mathbf{k}}$ in a quiescent fluid under creeping flow conditions is

$$\mathbf{u} = \frac{\Omega a^3}{r^2}\sin\theta\mathbf{e}_\phi$$

(d) Hence show that the torque acting on the sphere has magnitude $8\pi\mu\Omega a^3$.

Question 3

Viscous fluid occupies the region $0 < z < h$ between two rigid discs of radius a lying in the planes $z = 0$ and $z = h$, where $h \ll a$. The lower plane is at rest; the upper plane rotates about a vertical axis with constant angular velocity Ω.

(a) Scale the Navier-Stokes equations (in cylindrical coordinates) using h as the length scale for z, a as the length scale for σ, the obvious velocity scale and a viscous pressure scale to show that if the Reynolds number $\frac{\Omega h^2}{\nu} \ll 1$ the equations decouple and you might expect a solution of the form

$$\mathbf{u} = \sigma\Omega f(z)\mathbf{e}_\phi$$

(b) Hence find $\mathbf{u}$ and show that the torque on the upper disc required to sustain the flow is $\pi\mu\Omega a^4/h$, if end effects for $\sigma > a$ are neglected.

Question 4

Two model problems showing similar features to the Stokes equation/Oseen equations.

i. Consider the equation

$$y'' + (1/x)y' + \epsilon y y' = 0;\; y(1) = 0;\; y(\infty) = 1$$

Show that a regular expansion for $\epsilon \ll 1$ fails at the first term (this is like Stokes paradox).

ii. Consider the equation

$$y'' + (2/x)y' + \epsilon y y' = 0;\; y(1) = 0;\; y(\infty) = 1$$

Show that a regular expansion for $\epsilon \ll 1$ fails at the second term (this is like Whitehead's paradox).

Matched asymptotic expansions can also be used on these problems.

Question 5

Extend the problem done in lectures

$$\epsilon y'' + y' + y = 0, y(0) = 0, y(1) = 1$$

to $O(\epsilon)$ by the following procedure.

i. Calculate two terms of the outer solution i.e. $y^{out} = y_0 + \epsilon y_1 + O(\epsilon^2)$, and two terms of the inner solution.

ii. Match in the following way: rewrite y^{out} in the inner variable X, expand for small ϵ and keep only two terms - this gives $(y^{out})^{in}$ to $O(\epsilon^2)$.

Similarly for y^{in} - rewrite in the outer variable x, expand for small ϵ to $O(\epsilon^2)$ - this gives $(y^{in})^{out}$ to $O(\epsilon^2)$. Matching these determines all the constants in y^{in}.

iii. Hence construct a composite solution

$$y^c = y^{in} + y^{out} - (y^{in})^{out} = \exp(1-x) - (1+x)\exp(1-x/\epsilon) + \epsilon[(1-x)\exp(1-x) - \exp(1-x/\epsilon)]$$

Compare graphically/numerically with the exact solution for some suitable ϵ, say $\epsilon = 0.05$.

Question 6

Use matched asymptotic expansions to solve to O(1)

i.

$$\epsilon y'' + (1+\epsilon)y' + y = 0, y(0) = 0, y(1) = 1$$

This problem is similar to (actually a bit easier than) the problem done in lectures.

ii.

$$\epsilon y'' + 2y' + e^y = 0, y(0) = 0, y(1) = 0$$

a nonlinear boundary value problem

In each case,

i. Calculate the first term of the outer solution i.e. $y^{out} = y_0 + O(\epsilon)$, and the first term of the inner solution (the inner variable is $X = x/\epsilon$ in both cases).

ii. Match as in Question 5 but only keeping terms to $O(1)$.

iii. Hence construct a composite solution

$$y^c = y^{in} + y^{out} - (y^{in})^{out}$$

For the first problem,

$$y^c = y^{in} + y^{out} - (y^{in})^{out} = \exp(1-x) - \exp(1-x/\epsilon)$$

For the second problem, I don't know any exact solution, but you should get

$$y^c = y^{in} + y^{out} - (y^{in})^{out} = \ln(\frac{2}{1+x}) - \ln 2\; e^{-2x/\epsilon}$$