The University of Melbourne
Semester 1 Assessment 2002

Department of Mathematics and Statistics
620-361 OR Techniques and Algorithms
7 June 2002

Exam Duration: **3** *hours.*
Reading Time: **15** *minutes.*
The exam paper has **6** *pages.*
There are **2** *extra pages consisting of statistical normal tables.*

Authorized Materials:

Nonprogrammable calculators.
One (1) double-sided A4 formula sheet, hand written or typeset in 10 point or larger script.

Instructions to Invigilators:

The exam paper is to remain in the exam room.

Instructions to Students:

Make sure to answer the questions asked; points will not be given for answers to other questions! Total percentages for each question are given at the start of the question.

The University of Melbourne

ANNUAL EXAMINATION: JUNE 2002

620-361 OR TECHNIQUES AND ALGORITHMS

TIME ALLOWED: 3 HOURS

1) (15%, about 27 minutes work) Consider the unconstrained nonlinear program

$$\min_{x\in\mathbb{R}^2} f(x) = \frac{4}{3}x_1^3 - x_1x_2^2 - 8x_2 + 3x_2^2.$$

i) (2%) Show that $x = (1,2)^T$ is a stationary point of f.

Solution:

$$\nabla f(x) = \begin{bmatrix} 4x_1^2 - x_2^2 \\ -2x_1x_2 - 8 + 6x_2 \end{bmatrix}$$

so

$$\nabla f(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 4(1)^2 - (2)^2 \\ -2(1)(2) - 8 + 6(2) \end{bmatrix} = \begin{bmatrix} 4-4 \\ -4-8+12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and thus $x = (1,2)^T$ must be a stationary point of f.

ii) (3%) Does the second-order sufficiency condition hold for the stationary point $x = (1,2)^T$? Is $x = (1,2)^T$ a local minimum of f? Briefly justify your answer.

Solution: $f \in C^2$ since it involves only polynomial functions. Now

$$\nabla^2 f(x) = \begin{bmatrix} 8x_1 & -2x_2 \\ -2x_2 & -2x_1 + 6 \end{bmatrix}$$

so

$$\nabla^2 f(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 8 & -4 \\ -4 & 4. \end{bmatrix}$$

This has top left entry $8 > 0$ and determinant $16 > 0$ and so is positive definite, thus a second-order sufficiency condition does hold for $(1,2)^T$. Hence $(1,2)^T$ is a local minimum.

iii) (2%) Suppose $\{x^k\}$ converges to the point $(1,2)^T$, where $\{x^k\}$ is generated by the "simple Newton method". What rate of convergence do you expect? Briefly justify your answer.

Solution: As already noted f consists of polynomial functions so $f \in C^3$. Furthermore, we have already seen that $\nabla^2 f((1,2)^T)$ is positive definite, so a quadratic rate of convergence is expected.

iv) (1%) Find the direction of steepest descent for f at the point $x^0 = (2,0)^T$.
Solution: The steepest descent direction is

$$-\nabla f((2,0)^T) = -\begin{bmatrix} -4(2)^2 + (0)^2 \\ 2(2)(0) + 8 - 6(0) \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \end{bmatrix}$$

v) (4%) Find the Newton direction for f at $x^0 = (2,0)^T$. Is the Newton direction a descent direction? Justify your answer.
Solution: The Newton direction is given by

$$-\nabla^2 f((2,0)^T)^{-1}\nabla f((2,0)^T) = \begin{bmatrix} 16 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -16 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -16 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

This is a descent direction since

$$\langle \nabla f((2,0)^T), (-1,4)^T \rangle = \langle (16,-8)^T, (-1,4)^T \rangle = -16 - 32 = -48 < 0.$$

Alternatively, it is a descent direction since $\nabla^2 f((2,0)^T)$ is positive definite (it has positive eigenvalues 16 and 2).

vi) (3%) Show that a step of length $t_0 = 1$ along the Newton direction d at x^0 satisfies the Armijo-Goldstein condition, with the linesearch parameter $\sigma = \frac{1}{9}$.
Solution: We have

$$f(x^0 + t_0 d) = f((1,4)^T) = \frac{4}{3} - (4)^2 - 8(4) + 3(4)^2 = \frac{4}{3} - 16 - 32 + 48 = \frac{4}{3}$$

and

$$f(x^0) = f((2,0)^T) = \frac{4}{3}(2)^3 - 0 - 0 + 0 = \frac{32}{3}$$

as well as

$$\langle \nabla f(x_0), d \rangle = -48$$

from above. Now

$$f(x_0) + t_0\sigma\langle \nabla f(x_0), d \rangle = \frac{32}{3} + 1\frac{1}{9}(-48) = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

so the Armijo-Goldstein condition is satisfied if

$$f(x^0 + t_0 d) = \frac{4}{3} \leq \frac{16}{3} = f(x^0) + t_0\sigma\langle \nabla f(x^0), d \rangle.$$

Obviously, this holds, so the A-G condition is satisfied.

2) (14%, about 25 minutes work) Consider the nonlinear program (P):

$$\min_{x \in \mathbb{R}^2} \quad f(x) = (x_1+2)^2 + (x_2-2)^2$$
$$\text{s.t.} \qquad x_1 x_2 \geq 3 \quad \ldots\ldots\ldots (1)$$
$$2x_1 + x_2 \geq 5 \quad \ldots\ldots\ldots (2)$$

i) (5%) Write down the Lagrangian function for (P). Write down the KKT conditions for (P). Show that the point $x^* = (1,3)^T$ is a stationary point for (P), i.e. determine λ^* such that (x^*, λ^*) is a KKT point for (P).
Solution:

$$L(x,\lambda) = (x_1+2)^2 + (x_2-2)^2 + \lambda_1(3 - x_1x_2) + \lambda_2(5 - 2x_1 - x_2)$$

<u>KKTa.</u> $\nabla_x L(x,\lambda) = 0$, i.e. $2(x_1+2) - \lambda_1 x_2 - 2\lambda_2 = 0$ and $2(x_2-2) - \lambda_1 x_1 - \lambda_2 = 0$.
<u>KKTb.</u> (1) and (2) hold, $\lambda \geq 0$, $\lambda_1(3 - x_1x_2) = 0$ and $\lambda_2(5 - 2x_1 - x_2) = 0$.
<u>KKTc.</u> There are no equality constraints, so this condition does not apply.
x^* satisfies (1) and (2) at equality, so it must be that the first and last parts of KKTb hold. To show x^* is a stationary point for (P), we need to find $\lambda^* \geq 0$ such that KKTa holds, i.e. such that $2(1+2) - \lambda_1^*(3) - 2\lambda_2^* = 0$ and $2(3-2) - \lambda_1^*(1) - \lambda_2^* = 0$, i.e. such that $3\lambda_1^* + 2\lambda_2^* = 6$ and $\lambda_1^* + \lambda_2^* = 2$. A (unique) solution is $\lambda^* = (2,0)^T \geq 0$, so clearly $((1,3)^T, (2,0)^T)$ is a KKT point for (P) and $(1,3)^T$ is a stationary point.

ii) (3%) Which constraints are active at x^*? Explain briefly why a constraint qualification holds at x^*.
Solution: $x_1^* x_2^* = 1(3) = 3$ and $2x_1^* + x_2^* = 2 + 3 = 5$ so both constraints (1) and (2) are active. The constraint gradients for the active constraints are $\nabla g_1(x) = (-x_2, -x_1)^T$ and $\nabla g_2(x) = (-2,-1)^T$, respectively, so

$$\nabla g_1(x^*) = (-3,-1)^T \text{ and } \nabla g_2(x^*) = (-2,-1)^T.$$

It is clear these are linearly independent:

$$\begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix}$$

which is in row echelon form, and has full row rank. So the Linear Independence CQ holds. Alternatively, we seek d such that $(-3,-1)^T d < 0$ and $(-2,-1)^T d < 0$: obviously $d = (1,0)^T$ will do, so the Mangasarian-Fromowitz CQ holds.

iii) (6%) What is the critical cone for (P) at the point (x^*, λ^*)? State a second-order sufficiency condition for x^* to be a local minimum of (P). Does the second-order sufficiency condition hold?

Solution: $\lambda_1^* > 0$ and constraint (2) is active with $\lambda_2^* = 0$, so the critical cone is

$$\begin{aligned}\mathcal{C}(x^*, \lambda^*) &= \{d \in \mathbb{R}^2 \ : \ \langle (-3,-1)^T, d\rangle = 0, \ \langle (-2,-1)^T, d\rangle \leq 0\} \\ &= \{d \in \mathbb{R}^2 \ : \ 3d_1 + d_2 = 0, \ 2d_1 + d_2 \geq 0\} \\ &= \{d \in \mathbb{R}^2 \ : \ d_2 = -3d_1, \ d_1 \leq 0\}.\end{aligned}$$

The Hessian of the Lagrangian function is

$$\nabla^2_{xx} L(x, \lambda) = \begin{bmatrix} 2 & -\lambda_1 \\ -\lambda_1 & 2 \end{bmatrix}$$

so at the KKT point we have

$$\nabla^2_{xx} L(x^*, \lambda^*) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

A second-order sufficiency condition is that

$$d^T \nabla^2_{xx} L(x^*, \lambda^*) d > 0$$

for all $d \in \mathcal{C}(x^*, \lambda^*)$, $d \neq 0$, i.e. that

$$\begin{aligned}(d_1, -3d_1) \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ -3d_1 \end{bmatrix} = (d_1, -3d_1) \begin{bmatrix} 8d_1 \\ -8d_1 \end{bmatrix} &= 8d_1^2 + 24d_1^2 \\ &= 32d_1^2 > 0\end{aligned}$$

for all $d_1 \leq 0$. Obviously, this is the case, so the second-order sufficiency condition holds and x^* is a local minimum of (P).

3) (14%, about 25 minutes work)
Consider the nonlinear program

$$\min_{x \in \mathbb{R}^2} \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 \geq 0, \ x_2 \geq 2.$$

i) (5%) Write down the ℓ_2-penalty function $P_k(x)$ with penalty parameter k. Simplify $P_k(x)$ when $x_1 > 0$ and $x_2 < 2$. Write down $\nabla P_k(x)$ when $x_1 > 0$ and $x_2 < 2$.
Solution:

$$P_k(x) = \frac{1}{4}{x_1}^4 - \frac{1}{2}{x_1}^2 + {x_2}^2 + \frac{k}{2}\left(((2 - x_2)_+)^2 + ((-x_1)_+)^2\right)$$

When $x_1 > 0$, $(-x_1)_+ = 0$ and when $x_2 < 2$, $(2 - x_2)_+ = 2 - x_2$. So when $x_1 > 0$ and $x_2 < 2$, we have that

$$P_k(x) = \frac{1}{4}{x_1}^4 - \frac{1}{2}{x_1}^2 + {x_2}^2 + \frac{k}{2}\left(((2 - x_2))^2 + (0)^2\right) = \frac{1}{4}{x_1}^4 - \frac{1}{2}{x_1}^2 + {x_2}^2 + \frac{k}{2}(2 - x_2)^2.$$

and hence

$$\nabla P_k(x) = \begin{bmatrix} x_1^3 - x_1 \\ 2x_2 - k(2 - x_2) \end{bmatrix} = \begin{bmatrix} x_1^3 - x_1 \\ (2 + k)x_2 - 2k \end{bmatrix}.$$

ii) (4%) Let $x^k = (1, \frac{2k}{2+k})$ for each positive integer k. Verify that x^k is a stationary point of P_k such that $x_1^k > 0$ and $x_2^k < 2$. Write down the limit $x^* = \lim_{k\to\infty} x^k$.

Solution: Firstly we check $x_1^k = 1 > 0$, and $x_2^k = 2 - \frac{4}{2+k} < 2$ since $k > 0$. So $x_1^k > 0$, and $x_2^k < 2$. Furthermore

$$\nabla P_k(x^k) = \begin{bmatrix} (1)^3 - (1) \\ (2 + k)\frac{2k}{2+k} - 2k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so x^k is a stationary point as required. The limit

$$x^* = \lim_{k\to\infty} x^k = \begin{bmatrix} \lim_{k\to\infty} 1 \\ \lim_{k\to\infty} \frac{2k}{2 + k} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

iii) (5%) Write down an estimate λ^k of the optimal multiplier vector, and find the limit $\lambda^* = \lim_{k\to\infty} \lambda^k$.

Solution:

$$\left(g_1(x^k)\right)_+ = \left(-x_1^k\right)_+ = (-1)_+ = 0$$

so estimate

$$\lambda_1^k = k\left(g_1(x^k)\right)_+ = 0.$$

Also

$$\left(g_2(x^k)\right)_+ = \left(2 - x_2^k\right)_+ = (\frac{4}{2 + k})_+ = \frac{4}{2 + k}$$

so estimate

$$\lambda_2^k = k\left(g_2(x^k)\right)_+ = \frac{4k}{2 + k}.$$

The limit

$$\lambda^* = \lim_{k\to\infty} \lambda^k = \begin{bmatrix} \lim_{k\to\infty} 0 \\ \lim_{k\to\infty} \frac{4k}{2 + k} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

4) (7%, about 13 minutes work) Consider the nonlinear program (P) below, where log denotes the natural logarithm.

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & f(x) = (x_1-3)^2 + x_2^2 \\ \text{s.t.} \quad & \log x_1 - x_2 + 2 \le 0 \quad \ldots\ldots\ldots (1) \\ & -4x_1 - x_2 + 5 \le 0 \quad \ldots\ldots\ldots (2). \end{aligned}$$

i) (4%) Write down the Penalized Linear Program with Trust Region approximation to (P) at the point $x^0 = (1,2)^T$, i.e. write down PLPT(x^0). Use penalty parameter $\mu = 10$ and trust region parameter $\delta^0 = 1$. Do *not* attempt to solve the resulting LP.

Solution:

$$\nabla f(x) = \begin{bmatrix} 2(x_1-3) \\ 2x_2 \end{bmatrix} \quad \Rightarrow \quad \nabla f((1,2)^T) = \begin{bmatrix} 2(-2) \\ 2(2) \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

The second constraint is already linear, so we consider the first constraint. Let $g_1(x) = \log x_1 - x_2 + 2$. Then $g_1((1,2)^T) = 0 - 2 + 2 = 0$ and

$$\nabla g_1(x) = \begin{bmatrix} \frac{1}{x_1} \\ -1 \end{bmatrix} \quad \Rightarrow \quad \nabla g_1((1,2)^T) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so

$$\langle \nabla g_2((1,2)^T), (1,2)^T \rangle = (1,-1)(1,2)^T = -1.$$

Thus the linearized form of $g_1(x)$ at $(1,2)^T$ is

$$\nabla g_2((1,2)^T)x + g_2((1,2)^T) - \nabla g_1((1,2)^T)(1,2)^T = (1,-1)x + 0 - (-1) = x_1 - x_2 + 1.$$

The PLPT(x^0) approximation to (P) at $x^0 = (1,2)^T$ is as follows.

$$\begin{aligned} \min_{x\in\mathbb{R}^2, y\in\mathbb{R}} \quad & -4x_1 + 4x_2 + 10y \\ \text{s.t.} \quad & x_1 - x_2 + 1 \le y \\ & -4x_1 - x_2 + 5 \le 0 \\ & y \ge 0 \\ & 0 \le x_1 \le 2 \\ & 1 \le x_2 \le 3 \end{aligned}$$

ii) (1%) The simplex method applied to solve PLPT(x^0) in part (i) finds an optimal solution with optimal objective value 4 and optimal x values given by $\hat{x} = (2,3)^T$. Show that there must also be an optimal solution with x values $(1,2)^T$.

Solution: $(1,2)^T$ is feasible for the constraints of PLPT(x^0) so setting $y = 0$ we have a feasible solution with value $-4(1) + 4(2) + 10(0) = 4$. Since this is the optimal value, x^0 itself must be optimal

iii) (2%) In part (ii), you deduced that PLPT(x^0), approximating (P) at the point $x^0 = (1,2)^T$ has an optimal solution with $x = x^0$. What can you deduce about x^0 with respect to the nonlinear program (P)? Briefly justify your answer.
Solution: The point $x^0 = (1,2)^T$ is feasible for the NLP (P), since $g_1(x^0) = \log 1 - 2 + 2 = 0$ and $g_2(x^0) = -4(1) - 2 + 5 = -1 \leq 0$, so both constraints are satisfied. Therefore, since x^0 solves PLPT(x^0) with $\delta^0 > 0$, x^0 solves LP(x^0), and hence must be a stationary point for (P).

5) (18%, about 32 minutes work)

(a) (9%) Consider the nonlinear program (NLP):

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad \forall i = 1, \ldots, m. \end{aligned}$$

Prove that if (NLP) is a convex program with KKT point (x^*, λ^*), then x^* minimizes the Lagrangian function $L(x, \lambda^*)$, over all $x \in \mathbb{R}^n$, i.e.,

$$L(x^*, \lambda^*) \leq L(x, \lambda^*)$$

for all $x \in \mathbb{R}^n$. [Hint: Observe that $L(x, \lambda^*)$ is a function of x, show that x^* is a stationary point of this function, and explain briefly why the function is convex.]
Solution: By KKTa for (NLP), $\nabla_x L(x^*, \lambda^*) = 0$, so x^* is a stationary point of $L(x, \lambda^*)$. (NLP) convex implies $f(x)$ is convex and $g_i(x)$ is convex for each i. By KKTb, $\lambda_i^* \geq 0$, so $\lambda_i^* g_i(x)$ is also convex. Since the sum of convex functions is also convex it must be that $\sum_{i=1}^{m} \lambda_i^* g_i(x)$ is convex, and also $f(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x)$, since f is convex. But this is just $L(x, \lambda^*) = f(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x)$, by definition, so $L(x, \lambda^*)$ is a convex function of x. Now x^* a stationary point of a convex function means x^* is a global minimum, i.e.

$$L(x^*, \lambda^*) \leq L(x, \lambda^*)$$

for all $x \in \mathbb{R}^n$, as required.

(b) (9%) Consider the program (LP):

$$\begin{array}{lrl} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$.

(i) (2%) Explain briefly why (LP) is a convex program. Write down the Lagrangian function for (LP), using the equality constraint in the form $h(x) = Ax - b = 0$.

Solution: The objective function and all constraints are affine, and so (LP) is a convex program. The Lagrangian function is

$$L(x, \lambda, \eta) = c^T x - \lambda^T x - \eta^T (Ax - b).$$

(ii) (4%) Explain why the Lagrangian dual objective function for (LP) can be given by

$$\psi(\lambda, \eta) = \begin{cases} \eta^T b & \text{if } c^T - \lambda^T - \eta^T A = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Solution: The dual objective is

$$\begin{aligned} \psi(\lambda, \eta) &= \min_{x \in \mathbb{R}^n} L(x, \lambda, \eta) \\ &= \min_{x \in \mathbb{R}^n} \left(c^T x - \lambda^T x - \eta^T (Ax - b) \right) \\ &= \min_{x \in \mathbb{R}^n} \left((c^T - \lambda^T - \eta^T A)x + \eta^T b \right) \\ &= \eta^T b + \min_{x \in \mathbb{R}^n} (c^T - \lambda^T - \eta^T A)x. \end{aligned}$$

Now if any component of $c^T - \lambda^T - \eta^T A$ is negative, we can increase the corresponding component of x indefinitely, resulting in a value of $-\infty$. Similarly, if any component of $c^T - \lambda^T - \eta^T A$ is positive, we can decrease the corresponding component of x indefinitely, again resulting in a value of $-\infty$. Thus either $c^T - \lambda^T - \eta^T A = 0$ or the value of the dual objective function is $-\infty$.

(iii) (2%) Write down the Lagrangian dual problem for (LP), and simplify if possible.

Solution:

$$\max_{\lambda \geq 0, \eta} \psi(\lambda, \eta) = \begin{cases} \max_{\lambda \geq 0, \eta} & \eta^T b \\ \text{s.t.} & c^T - \lambda^T - \eta^T A = 0. \end{cases}$$

(iv) (1%) Given $(\hat{\lambda}, \hat{\eta})$ with $\hat{\lambda} \geq 0$ and $c^T - \hat{\lambda}^T - \hat{\eta}^T A = 0$, what can you deduce about the optimal value of (LP)?

Solution: Since (LP) is convex, by weak duality it must be that the optimal value of (LP) is at least $\hat{\eta}^T b$.

6) (7%, about 13 minutes work) A small-scale jam-making company is planning its production over the next planning period. The company makes strawberry and raspberry jam, for which sugar is a key ingredient. The sweetness of the fruit available varies, so that the amount of sugar required to make the jam also varies. On average, 100g of sugar are required per bottle of strawberry jam, and 150g of sugar are required per bottle of raspberry jam. The number of grams of sugar per bottle required to make strawberry jam and raspberry jam is well modelled by a joint normal distribution, with covariance matrix

$$S = \begin{bmatrix} 75 & 10 \\ 10 & 90 \end{bmatrix}.$$

In the next planning period, 500,000g of sugar are available.

i) (3%) The company wants to plan its production, i.e. the number of bottles of strawberry jam and the number of bottles of raspberry jam to be made, to ensure the probability that the sugar available is sufficient for production during the period is at least 0.99. Formulate this requirement as a chance constraint.

Solution: Let

$$\begin{aligned} J_S = & \text{ random variable for grams of sugar needed per bottle of strawberry jam, and} \\ J_R = & \text{ random variable for grams of sugar needed per bottle of raspberry jam} \end{aligned}$$

and let x_S denote the number of bottles of strawberry jam to make, and x_R the number of bottles of raspberry jam to make, in the coming period. Then the chance constraint is

$$Pr(J_S x_S + J_R x_R \leq 500000) \geq 0.99.$$

ii) (4%) Model the chance constraint problem in part (i) as a nonlinear constraint on the number of bottles of strawberry and raspberry jam to be made. Say briefly why you would expect the resulting constraint function to be convex.

Solution: We let J = random variable for the number of grams of sugar used, so $J = J_S x_S + J_R x_R$. Then J is normally distributed with mean $\mu_J = 100x_S + 150x_R$ and variance

$$\sigma_J^2 = [x_S, x_R] \begin{bmatrix} 75 & 10 \\ 10 & 90 \end{bmatrix} \begin{bmatrix} x_S \\ x_R \end{bmatrix} = 75x_S^2 + 20x_S x_R + 90x_R^2.$$

Thus

$$
\begin{aligned}
& & Pr(J_S x_S + J_R x_R \le 500000) & \ge 0.99 \\
\Leftrightarrow & & Pr(J \le 500000) & \ge 0.99 \\
\Leftrightarrow & & Pr(Z \le \tfrac{500000-\mu_J}{\sigma_J}) & \ge 0.99 \\
\Leftrightarrow & & \tfrac{500000-\mu_J}{\sigma_J} & \ge 2.33 \\
\Leftrightarrow & & \mu_J + 2.33\sigma_J & \le 500000 \\
\Leftrightarrow & & 100x_S + 150x_R + 2.33\sqrt{75x_S^2 + 20x_S x_R + 90x_R^2} & \le 500000.
\end{aligned}
$$

For any covariance matrix S, $\sqrt{x^T Sx}$ is convex. Since $2.33 > 0$ and $100x_S + 150x_R$ is a linear and hence convex function, we have that

$$100x_S + 150x_R + 2.33\sqrt{75x_S^2 + 20x_S x_R + 90x_R^2} - 500000$$

is a convex function.

7) (10%, about 18 minutes work) A pre-fabricated swimming pool retailer sells 250 swimming pools per year. Delivery of new pools from the manufacturer costs \$2000 per order and must be ordered 3 weeks in advance. It is estimated that the cost of holding a pool in stock is \$5000 per year. The pool retailer is operates 50 weeks of the year. Stock-outs are permitted, but the cost to the retailer for any pool short is estimated to be \$100 per week.

i) (3%) What is the optimal number of swimming pools Q^* to reorder? What is the order cycle time?
Solution: We have $a = 250$ pools per year, $K = 2000$, $h = 5000$ and $s = 100(50) = 5000$. The optimal order quantity is

$$Q^* = \sqrt{\frac{2Ka(h+s)}{hs}} = \sqrt{\frac{2(2000)(250)(10000)}{25000000}} = \sqrt{400} = 20$$

pools. Cycle time is $Q^*/a = 20/250 = 0.08$ years $= 0.08(50) = 4$ weeks.

ii) (4%) What is the maximum number of pools to be backlogged each order cycle? How much storage space must the retailer have available?
Solution: The storage space used is that required for

$$M^* = \sqrt{\frac{2Kas}{h(h+s)}} = \sqrt{\frac{2(2000)(250)(5000)}{5000(10000)}} = \sqrt{100} = 10$$

pools. The maximum number of pools to be backlogged each cycle is

$$Q^* - M^* = 20 - 10 = 10$$

pools.

iii) (1%) What is the total cost of meeting demand per year?

Solution: Total cost of meeting demand is

$$s(Q^* - M^*) = 5000(10) = 50,000$$

dollars per year.

iv) (2%) Give a rule for when to place orders, based on the number of pools in the storage area.
Solution: Since exactly half the demand is backlogged, backlogging starts half-way through the inventory cycle, i.e., two weeks before the order arrives. Since the lead time is three weeks, we must order one week before the inventory level is zero. This is half-way through the period of non-negative stock levels, so the stock level must be half its maximum, i.e., we should reorder when there are 5 pools in the storage area.

8) (15%, about 27 minutes work) Consider the swimming pool retailer discussed in the previous question, but with annual demand better modelled by a normally distributed random variable, with mean 250 and variance 800. Assume that demand can be backlogged and that the cost to the retailer is \$5000 per pool short. Assume here that the lead time is only 1 week.

i) (3%) Estimate the number of pools that should be ordered, each time an order is placed.
Solution: We have demand D distributed according to $N(250, 800)$, $K = 2000$ and $h = 5000$. Each time an order is placed, the retailer should order

$$Q^* \approx \sqrt{\frac{2K\mathcal{E}(D)}{h}} = \sqrt{\frac{2(2000)(250)}{5000}} = \sqrt{200} \approx 14 \text{ pools.}$$

ii) (3%) Let the random variable X denote the number of pools that are ordered during the lead time. How is X distributed? What is its mean and standard deviation?
Solution: We have lead time $L = \frac{1}{50} = 0.02$ years. X is normally distributed according to $N(L\mathcal{E}(D), Lvar(D)) = N(0.02(250), 0.02(800)) = N(5, 16)$, so X has mean 5 pools and standard deviation of 4 pools.

iii) (5%) What equation involving X should the reorder point R^* satisfy? Using the attached table of cumulative standard normal probabilities, determine such a reorder point R^*.
Solution: We have $c_B = 5000$ so R^* should satisfy

$$Pr(X \geq R^*) = \frac{hQ^*}{c_B\mathcal{E}(D)} = \frac{5000(14)}{5000(250)} = 0.056.$$

Now

$$
\begin{aligned}
& & Pr(X \geq R^*) &= 0.056 \\
\Leftrightarrow & & Pr(Z \geq \tfrac{R^*-5}{4}) &= 0.056 \\
\Leftrightarrow & & 1 - Pr(Z \leq \tfrac{R^*-5}{4}) &= 0.056 \\
\Leftrightarrow & & Pr(Z \leq \tfrac{R^*-5}{4}) &= 1 - 0.056 = 0.944 \\
\Leftrightarrow & & \tfrac{R^*-5}{4} &\approx 1.59 \\
\Leftrightarrow & & R^* &\approx 4(1.59) + 5 = 11.36 \approx 11
\end{aligned}
$$

so the reorder point should be set to 11 pools.

iv) (1%) Give an expression for the safety stock level in terms of X and R^*. Now calculate the safety stock level for the pool retailer.
Solution: Safety stock is

$$R^* - \mathcal{E}(X) = 11 - 5 = 6.$$

v) (1%) What is the expected value of the maximum level of inventory, i.e. the inventory level at the start of each order cycle?
Solution: The expected maximum inventory level is

$$R^* - \mathcal{E}(X) + Q^* = 11 - 5 + 14 = 20.$$

vi) (2%) If the lead time had been normally distributed with a mean of 1 week, it is known that the variance of X would have increased. Would you expect this to increase or decrease the safety stock level? Explain in at most three lines.
Solution: The reorder point is given by $R^* = var(X)(1.59) + \mu_X$, so an increase in variance of X will increase the reorder point. The mean value of X, $\mathcal{E}(X)$, has not changed, since it will be the mean lead time by the mean demand, so the safety stock level, $R^* - \mathcal{E}(X)$ must increase.

END OF QUESTIONS

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