The University of Melbourne
Semester 1 Assessment 2003

Department of Mathematics and Statistics
620-361 OR Techniques and Algorithms
13 June 2003

Exam Duration: **3** *hours.*
Reading Time: **15** *minutes.*
The exam paper has **5** *pages.*

Authorized Materials:
Nonprogrammable calculators.

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ANNUAL EXAMINATION: JUNE 2003
620-361 OR TECHNIQUES AND ALGORITHMS

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Throughout this paper, log denotes the natural logarithm.

1 (a) The Fibonacci numbers F_n satisfy the recursion

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

with $F_0 = F_1 = 1$.

Show that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]. \tag{2}$$

(b) Now let $\gamma_n = F_{n-1}/F_n$. Use (1) to derive a quadratic equation for $\gamma \equiv \lim_{n\to\infty} \gamma_n$. (You may assume that this limit exists.)

(c) Solve the equation in (b) and verify that there is a solution in $[0, 1]$ that coincides with $\lim_{n\to\infty} F_{n-1}/F_n$ derived from (2).

(d) Apply the golden section search to the function

$$f(x) = (x - \gamma)(x - 2\gamma) \tag{3}$$

on the interval $[0, 2]$ to obtain an interval of length 2γ that contains the minimum of f.

(e) Apply two iterations of Newton's Method to find the minimum of the function f defined in (3), starting at $x_0 = \gamma$.

(f) Explain the behaviour that you observed in part (e).

[24 marks]

2) NB: In the following question, you may use the fact that the eigenvalues of a matrix of the form

$$\begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}$$

are $a + 1$ and $a - 1$.

Consider the function

$$f(x_1, x_2) = -\left(e^{-x_1^2} + e^{-x_2^2} + x_1 x_2 \right). \tag{4}$$

(a) Using the first-order necessary conditions, show that $x^1 = (0,0)^T$ and $x^2 = (\sqrt{\log(2)}, \sqrt{\log(2)})^T$ are stationary points of f.

(b) Use the second-order sufficiency conditions to show that one of these points is a local minimum and the other is neither a local minimum nor a local maximum.

(c) At the point which is neither a local minimum nor a local maximum, give two directions d_1 and d_2 along which the function f increases and decreases respectively.

[12 marks]

3) Consider the function

$$f(x_1, x_2) = \frac{1}{3}x_1^3 - x_1 x_2^2 - 3x_2 + 3x_2^2. \tag{5}$$

(a) Find the direction of steepest descent for f at the point $x^0 = (1,1)^T$.

(b) Find the point x^1, arrived at after one iteration of the steepest descent algorithm.

(c) Show that the Armijo-Goldstein condition is satisfied for $\sigma = 1/2$ by the step that is taken in this iteration.

(d) The Wolfe condition is satisfied for all $\mu \in [1/2, 1]$ by this step. Explain why this is so.

(e) Find the Newton direction for f at x^0. Is the Newton direction a descent direction? Justify your answer.

[16 marks]

4) Consider the nonlinear program

$$\min_{x \in \mathbb{R}^2} f(x) = (x_1+1)^2 + (x_2+1)^2 \tag{6}$$

such that

$$x_1 x_2 \geq 4 \tag{7}$$
$$x_1 + x_2 \geq 3. \tag{8}$$

(a) Write down the Lagrangian function for (6). Show that the point $x^* = (2,2)^T$ is a stationary point for (6).

(b) Which constraints are active at x^*? Show that a constraint qualification holds at x^*.

(c) What is the critical cone for (6) at the point (x^*, λ^*)?

(d) State a second-order sufficiency condition for x^* to be a local minimum of (6). Does this condition hold?

[16 marks]

5) Consider the nonlinear program

$$\min_{x \in \mathbb{R}^2} x_1^4 - 2x_1^2 + \frac{3}{2}x_2^2 \quad \text{subject to} \quad x_1 \geq 0,\ x_2 \geq 3. \tag{9}$$

(a) Write down the ℓ_2-penalty function $P_k(x)$ with penalty parameter k.

(b) Find a stationary point x^k of $P_k(x)$ such that $x_1^k > 0$ and $x_2^k < 3$. Write down the limit $x^* = \lim_{k\to\infty} x^k$.

(c) Write down an estimate λ^k of the optimal multiplier vector, and find the limit $\lambda^* = \lim_{k\to\infty} \lambda^k$.

(d) Verify that x^* and λ^* satisfy the KKT conditions for the original problem (9).

[16 marks]

6) Consider an Asymmetric Traveling Salesman Problem (ATSP) on 50 nodes. This problem is the same as the TSP taught in class, except that the cost $C(i, j)$ of travel from city i to city j may differ from the cost $C(j, i)$ of travel from city j to city i. We wish to solve the problem using a simulated annealing algorithm.

(a) Given a particular tour $s = (s_1, \ldots, s_{50})$, would a 2-opt procedure be appropriate in obtaining neighbouring tours for the ATSP? Give your reasons.

A 3-opt procedure for obtaining neighbouring tours for the 50-node ATSP can be designed as follows.

Given a current tour $(s_1, \ldots, s_{50})$, a neighbouring tour is obtained by randomly selecting j, k, l, for $1 \leq j < k < l \leq 50$, and swapping arcs: (s_j, s_{j+1}), (s_k, s_{k+1}) and (s_l, s_{l+1}) with (s_j, s_{k+1}), (s_k, s_{l+1}) and (s_l, s_{j+1}). Let $t = (t_1, \ldots, t_{50})$ be the resulting neighbouring tour.

(b) Suppose the cost of the tour s is $f(s)$. Write down a formula to obtain the cost of t (as described above), without having to add up the costs of all arcs used in t from scratch.

(c) How many different tours are there in the 3-opt neighbourhood for a 50-node ATSP.

(d) The Aarts and Larhoven method is used to determine an initial temperature. Specifically, three tours π^i for $i = 1, 2, 3$ are selected at random. Let

1. N be the size of the neighbourhood,
2. n_i^+ be the number neighbouring tours of π^i that have cost strictly greater than that of π^i, and
3. μ_i^+ be the average difference in cost among those neighbouring tours of π^i that have cost strictly greater than that of π^i.

The following statistics were observed

- Tour π^1: $n_1^+/N = 0.42$, $\mu_1^+ = 245$;
- Tour π^2: $n_2^+/N = 0.65$, $\mu_2^+ = 341$;
- Tour π^3: $n_3^+/N = 0.52$, $\mu_3^+ = 299$.

Suppose that an initial acceptance rate of $\chi = 0.65$ is used, what is the value of the initial temperature that we should use for this SA algorithm?

[16 marks]

END OF QUESTIONS