# The University of Melbourne
# Semester 1 Assessment 2004

## Department of Mathematics and Statistics
## 620-361 OR Techniques and Algorithms
## 17 June 2004

*Exam Duration:* **3** *hours.*
*Reading Time:* **15** *minutes.*
*The exam paper has* **5** *pages.*

**Authorized Materials:**
Nonprogrammable calculators.

*The University of Melbourne*

ANNUAL EXAMINATION: JUNE 2004

620-361 OR TECHNIQUES AND ALGORITHMS

TIME ALLOWED: 3 HOURS

**1.** The Fibonacci numbers $F_n$ satisfy the recursion

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = F_1 = 1$.

**(a)** Show that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

**(b)** The Fibonacci search method, which is based on the Fibonacci numbers, can be used to approximate the minimum of a unimodal function of one variable on a closed interval. Give one reason why the Fibonacci search method is considered efficient, and briefly explain why the search that it performs is unbiased.

**(c)** State the role of the quantity $\gamma_n = F_{n-1}/F_n$ in the Fibonacci search method.

**(d)** Explain how the golden section search method is related to the Fibonacci search method, and hence derive the golden ratio

$$\gamma = \frac{\sqrt{5}-1}{2}.$$

**(e)** Apply the golden section search to the function

$$f(x) = (x-\gamma)(x-3\gamma) \qquad (1)$$

on the interval $[0, 3]$ to obtain an interval of length exactly $3\gamma$ that contains the minimum of $f$.

**(f)** Using the interval obtained in part (e), write down an estimate, $x^*_{est}$, of the minimum of the function $f$ defined in (1). Compare your estimate with the exact minimum, $x^*$, and write down the error, $|x^*_{est} - x^*|$, associated with your estimate. Suppose you did not know $x^*$, give an upper bound for the error associated with your estimate.

**(g)** Apply one iteration of Newton's Method (single variable case) to estimate the minimum of the function $f$ defined in (1), starting at $x_0 = \gamma$. What is the error associated with this estimate?

[21 marks]

**2.** Consider the quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x) = \frac{1}{2} x^T B x + c^T x,$$

where $c$ is a column vector and $B$ is a symmetric matrix.

**(a)** State a necessary condition for $f$ to have a unique stationary point.

**(b)** State a sufficient condition for $f$ to have a unique global minimum, giving reasons for your answer.

**(c)** Suppose $n = 2$, $c = (1, 1)^T$ and

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Is $f$ a convex function? Write down the set of all global minima of $f$, and justify your answer.

[10 marks]

**3.** Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1 - x_1 x_2.$$

**(a)** Using the first-order necessary conditions, show that $(0, 0)^T$ and $(\frac{2}{5}, \frac{1}{\sqrt{5}})^T$ are stationary points of $f$.

**(b)** Use the second-order sufficiency conditions to show that one of these points is a local minimum and the other is neither a local minimum nor a local maximum.

**(c)** Is the local minimum identified in part (b) also a global minimum? Justify your answer.

[12 marks]

**4.** Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x_1, x_2) = 2(x_1 - 3)^2 + (x_2 - 2)^2.$$

**(a)** Find the direction of steepest descent for $f$ at the point $x^0 = (1, 1)^T$.

**(b)** Show that $t^* = \frac{17}{66}$ is the stepsize which minimises $f$ in the direction of steepest descent at $x^0 = (1, 1)^T$. Hence find the point $x^1$, arrived at after one iteration of the steepest descent algorithm.

**(c)** Show that the stepsize given in part (b) satisfies the Armijo-Goldstein rule when the Armijo-Goldstein weight parameter $\sigma$ takes any value on the interval $(0, \frac{1}{2}]$.

**(d)** Find the Newton direction for $f$ at $x^0 = (1, 1)^T$. Is the Newton direction a descent direction? Justify your answer.

**(e)** Explain why only one iteration of the simple Newton method would be required to find the global minimum of $f$.

[18 marks]

**5.** Consider the constrained nonlinear program (NLP)

$$\min_{x \in \mathbf{R}^2} f(x) = -20x_1 - 10x_2$$

subject to

$$\begin{aligned} x_1^2 + x_2^2 &\leq 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

**(a)** Write down the Lagrangian for the NLP, and show that $(x^*, \lambda^*) = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 5\sqrt{5}, 0, 0)^T$ is a KKT point for the NLP.

**(b)** Which constraints are active at $x^*$? Does a constraint qualification hold at $x^*$ ?

**(c)** Identify the critical cone at the point $(x^*, \lambda^*)$.

**(d)** State a second-order sufficiency condition for $x^*$ to be a local minimum of the NLP. Show that this condition holds, and find the value of the objective function at $x^*$.

**(e)** Sketch the feasible region and lines of constant objective function. Use the sketch to argue that the point $x^*$ is the unique global minimum of the NLP.

[16 marks]

**6.** **(a)** Describe, in general terms, how the $\ell_2$-penalty method approximates a solution to a constrained nonlinear program.

**(b)** Consider the constrained nonlinear program

$$\min_{x \in \mathbb{R}^2} x_1^4 - x_1^2 + \frac{1}{2}x_2^2 \quad \text{subject to} \quad x_1 \geq 0,\ x_2 \geq 3. \tag{2}$$

**(i)** For the problem (2), write down the $\ell_2$-penalty function $P_k(x)$ with penalty parameter $k$.

**(ii)** Find a stationary point $x^k$ of $P_k(x)$ such that $x_1^k > 0$ and $x_2^k < 3$. Write down the limit $x^* = \lim_{k \to \infty} x^k$.

**(iii)** Write down an estimate $\lambda^k$ of the optimal multiplier vector, and find the limit $\lambda^* = \lim_{k \to \infty} \lambda^k$.

**(iv)** Verify that $x^*$ and $\lambda^*$ satisfy the KKT conditions for the original problem given in (2).

[16 marks]

**7.** Two data communications links, labelled $A$ and $B$, are available to carry data traffic from Melbourne to Sydney. Data traffic is generated in Melbourne at a rate of $d$ megabytes per second. All traffic must be routed to Sydney using one or both of the links $A$ and $B$. The cost associated with routing $x$ megabytes per second on any given link is given by $D(x)$, where

$D$ is a convex, strictly increasing and differentiable function.

Consider the task of optimising the system performance for this communications network. Specifically, we seek to minimise the total cost associated with carrying traffic from Melbourne to Sydney (note, there is no data traffic travelling in the opposite direction from Sydney to Melbourne).

**(a)** Formulate the problem as a constrained nonlinear program.

**(b)** Use the KKT conditions to deduce that if the optimal solution is such that both links are utilised, then the links must have equal *marginal cost* (that is, equal derivatives of the cost function) when the optimal solution is implemented.

**(c)** If the optimal solution is such that only link $A$ is utilised, what can be said about the marginal costs of links $A$ and $B$?

[7 marks]

## END OF QUESTIONS