The University of Melbourne
Semester 1 Assessment 2005

Department of Mathematics and Statistics
620-361 OR Techniques and Algorithms
22 June 2005

Exam Duration: **3** *hours.*
Reading Time: **15** *minutes.*

Authorized Materials:
Nonprogrammable calculators.

The University of Melbourne

ANNUAL EXAMINATION: JUNE 2005
620-361 OR TECHNIQUES AND ALGORITHMS

TIME ALLOWED: 3 HOURS

1. The Fibonacci numbers F_n satisfy the recursion

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = F_1 = 1$.

(a) The Fibonacci search method, which is based on the Fibonacci numbers, can be used to approximate the minimum of a unimodal function of one variable on a closed interval. Give one reason why the Fibonacci search method is considered efficient, and briefly explain why the search that it performs is unbiased.

(b) Briefly explain how the golden section search method is related to the Fibonacci search method.

(c) Suppose we wish to use the Fibonacci search method to estimate the minimum of the function

$$f(x) = (x - 2)^2 + 1$$

on the interval $[1, 4]$, to within a tolerance of 0.1. Using the fact that

$$F_6 = 13, \quad F_7 = 21,$$

state how many f-calculations will be required for this task. Perform *one* iteration of the Fibonacci search method (that is, one reduction of the length of the search interval).

(d) Apply one iteration of Newton's method (single variable case) to estimate the minimum of the function $f(x)$ defined in part (c), starting at $x = 1$. What is the error associated with this estimate? Briefly explain your answer.

[15 marks]

2. Consider the quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x) = \frac{1}{2}x^T Bx + c^T x$$

where c is a column vector and B is a symmetric matrix.

(a) Write down $\nabla f(x)$, and hence state a property of the matrix B which guarantees that f has a unique *stationary point*.

(b) State a property of the matrix B which guarantees that f has a unique global minimum, giving reasons for your answer.

(c) Suppose $n = 2$, $c = (1, 1)^T$ and

$$B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Find all global minima of f.

[9 marks]

3. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x_1, x_2) = x_1^2 x_2 - 4x_1 x_2 + x_2^3 + x_2$$

(a) Using the first-order necessary condition, show that

$$(2, 1)^T \text{ and } (2 - \sqrt{3}, 0)^T$$

are stationary points of f.

(b) Use the second-order sufficiency conditions to show that one of these points is a local minimum and the other is neither a local minimum nor a local maximum.

(c) At the point which is neither a local minimum nor a local maximum, give a direction d along which the function increases.

(d) Is the local minimum also a global minimum ? Justify your answer.

[12 marks]

4. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a twice continuously differentiable function, and let $x^0 \in \mathbf{R}^n$ denote an arbitrary initial point for a descent method.

(a) Give the definition of a descent direction for f at x^0.

(b) The second-order Taylor series approximation of f at x^0 is given by

$$Q(x) = f(x^0) + \nabla f(x^0)^T(x - x^0) + \frac{1}{2}(x - x^0)^T \nabla^2 f(x^0)(x - x^0)$$

Use the second-order Taylor series approximation to derive the Newton direction for f at x^0

$$-\nabla^2 f(x^0)^{-1} \nabla f(x^0)$$

where $\nabla^2 f(x^0)^{-1}$ denotes the inverse of $\nabla^2 f(x^0)$. Give a brief explanation of your derivation.

(c) Show that if $\nabla^2 f(x^0)$ is positive definite, then the Newton direction is a descent direction for f at x^0. *Note: you may use without proof the result which states that if A is a positive definite matrix, then A^{-1} exists and is also positive definite.*

(d) Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x) = x_1^2 + \frac{1}{2}x_2^2$$

Show that $t^* = 1$ is the stepsize which minimises f along the Newton direction for f at $x^0 = (1, 1)^T$. Hence perform one iteration of Newton's method with $x^0 = (1, 1)^T$. Comment on your result.

(e) Explain why more than one iteration of the steepest descent method would be required in order to find the global minimum of the function f given in part (d), when the initial point $x^0 = (1, 1)^T$ is used.

[18 marks]

5. Consider the constrained nonlinear program (NLP)

$$\min_{x \in \mathbf{R}^2} f(x) = (x_1 - 2)^2 + (x_2 - 2)^2$$

subject to

$$x_1 + x_2 = b$$

where $b \in \mathbf{R}$.

(a) Show that the NLP has a global minimum at $x^* = (\frac{b}{2}, \frac{b}{2})^T$.

(b) State a physical interpretation of the Lagrange multiplier associated with the equality constraint of the NLP. Give a linear estimate of the change in the objective function value as a function of change in the right-hand side of the equality constraint, and hence estimate the objective function value if this constraint is changed to

$$x_1 + x_2 = b + \frac{1}{10}$$

[7 marks]

6. Consider the constrained nonlinear program (NLP)

$$\min_{x \in \mathbf{R}^2} f(x) = -x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1^2 + x_2^2 &\leq 1 & (1) \\ x_1 &\geq 0 & (2) \\ x_2 &\geq 0 & (3) \end{aligned}$$

(a) Let λ_1, λ_2 and λ_3 be the Lagrange multipliers associated with constraints (1), (2) and (3), respectively. Write down the Lagrangian for the NLP. Show that $(x^*, \lambda^*) = (1, 0, \frac{1}{2}, 0, 2)^T$ is a KKT point for the NLP.

(b) Which constraints are active at x^*? Does a constraint qualification hold at x^* ?

(c) Show that the critical cone at the point (x^*, λ^*) is given by

$$\{d \in \mathcal{R}^2 : d_1 = 0, d_2 = 0\}$$

(d) State a second-order sufficiency condition for x^* to be a local minimum of the NLP, and hence deduce that (x^*, λ^*) is a local minimum.

(e) Find the value of the objective function at x^*. Sketch the feasible region and lines of constant objective function. Use the sketch to argue that the point x^* is the unique global minimum of the NLP.

(f) In terms of sensitivity analysis, what qualitative information can we obtain from the fact that $\lambda_1^* > 0$, $\lambda_3^* > 0$, and $\lambda_2^* = 0$?

[16 marks]

7. (a) Describe, in general terms, how the ℓ_2-penalty method approximates a solution to a constrained nonlinear program.

(b) Consider the constrained nonlinear program

$$\min_{x \in \mathbb{R}^2} x_1^4 - x_1^2 + \frac{1}{2} x_2^2 \quad \text{subject to} \quad x_1 \geq 0, \ x_2 \geq 4. \tag{4}$$

(i) For the problem (4), write down the ℓ_2-penalty function $P_k(x)$ with penalty parameter k.

(ii) Find a stationary point x^k of $P_k(x)$ such that $x_1^k > 0$ and $x_2^k < 4$. Write down the limit $x^* = \lim_{k \to \infty} x^k$.

(iii) Write down an estimate λ^k of the optimal multiplier vector, and find the limit $\lambda^* = \lim_{k \to \infty} \lambda^k$.

(c) What difficulty arises if we attempt to employ second-order search methods, such as Newton's method, in order to estimate the minimum of the penalty function $P_k(x)$?

[16 marks]

8. Let x^* be an unconstrained local minimum of a function $f : \mathbf{R}^n \to \mathbf{R}$, and assume that f is continuously differentiable. Prove that $\nabla f(x^*) = 0$ (first order necessary condition).

[7 marks]

END OF QUESTIONS