620-361 Operations Research Techniques and Algorithms

Semester 1 Exam — June, 2007

DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF MELBOURNE

Exam duration: three hours
Reading time: fifteen minutes
This paper has four (4) pages–including this page

Authorised materials:
Non-programmable calculators are permitted.

Instructions to invigilators:
The exam paper may be taken out of the examination room.

Instructions to students:
There are seven (7) questions. All questions may be attempted.
The number of marks for each question is indicated.
The total number of marks available is 100
Useful formulae are provided on page 4.

This paper is to be lodged with the Baillieu Library

PROBLEM 1 [4 marks]

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a unimodal function. We wish to use a numerical line search method to estimate its minimum on the closed interval $[a, b]$, with a pre-specified error tolerance of ϵ. Explain the similarities and the differences between the Fibonacci and the Golden Section search methods. Include an argument on the number of iterations and expected execution time.

PROBLEM 2 [8 marks]

Consider the problem of minimising the function

$$f(x) = \begin{cases} -x^2 - 2 & x < 2 \\ x^2 - 10 & x \geq 2 \end{cases}$$

over the interval $[1, 5]$.

(a) [2 MARKS] Perform one iteration of Newton's method starting at $x^0 = 1.5$ and show that it has converged.

(b) [2 MARKS] Perform one iteration of Newton's method starting at $x^0 = 3$ and show that it has converged.

(c) [4 MARKS] Argue by inspection (plot) that $x^* = 2$ and explain why Newton's method fails to converge to the optimal value, and explain the behaviour of the algorithm for each of the two initial points above.

PROBLEM 3 [8 marks]

Consider the function $f(x) = \frac{4}{3}x_1^3 - 2x_1x_2 + \frac{1}{2}x_2^2$.

(a) [2 MARKS] Show that the point $x^* = (1, 2)^T$ is a stationary point.

(b) [6 MARKS] Use the second order condition to show that it is a local minimum. Is it a *global* minimum? Explain your answer.

PROBLEM 4 [20 marks]

Consider the problem of minimising $f(x) = x_1^2 + x_2^2 - x_1x_2 - 3x_1 + 3x_2 + 3$.

(a) [4 MARKS] Find all stationary points and determine the global minimum.

(b) [8 MARKS] Perform one step of the steepest descent method starting at $x^0 = (1, 1)^T$. Identify the direction d and show that the function $q(t) = f(x^0 + td)$ to be minimised is $q(t) = 4 - 20t + 28t^2$. Then calculate x^1.

(c) [8 MARKS] Verify that Newton's direction is a descent direction at $x^0 = (1, 1)^T$ and perform one step of Newton's method.

PROBLEM 5 [16 marks]
Let $f\colon \mathbb{R}^n \to \mathbb{R}, f \in C^3$. Consider the non-linear problem $\min f(x)$, subject to $h(x) = c^T x - b = 0$. Let x^* be a stationary point for this problem.

(a) [4 MARKS] At x^*, the feasible directions d are such that for sufficiently small t, $h(x + td) = 0$. Show that for this problem, the feasible directions satisfy $\nabla h(x^*)^T d = 0$.

(b) [12 MARKS] Suppose that $\nabla^2 f(x^*)$ is full rank and has eigenvalues $\lambda_1 < 0 < \lambda_2 < \dots \lambda_n$. Show that if $c = Cv_1$, for a non-zero constant C and v_1 the eigenvector with (negative) eigenvalue λ_1, then x^* is a local minimum.

PROBLEM 6 [24 marks]
This problem is typical of economics modelling. A factory produces machines that can be sold for \$ 270 each. If K units of capital per week and L hours of labour per week are used, then $L^{1/2}K^{1/3}$ machines are produced. One unit of capital costs \$5. A total of 160 hours of labour per week are available at \$10/hour through the payroll personnel. Additional labour can be hired externally at a cost of \$25/hour. Model the optimisation problem that maximises profit (revenue minus costs). It can be shown that none of the non-negativity constraints for the hours of labour and for the capital are active at the optimum. By exploring the KKT conditions for stationarity, calculate the value of the Lagrange multipliers at the optimum, and argue that all 160 hours of cheaper labour are used at the optimum. How much profit can the company make buying an extra hour of cheap labour? Based on your answer, would you recommend the company increase, decrease or keep the same number of permanent personnel? [*Notice that you are not asked to solve this problem completely.*]

PROBLEM 7 [20 marks]
Consider the problem:

$$\begin{aligned} \min f(x) &= 3x_1^2 + 4x_1x_2 \\ \text{s.t.}\ \ & x_1 + 2x_2 - 3 \le 0 \end{aligned}$$

(a) [4 MARKS] Show that $x^* = (-3, 3)^T$ is a KKT point for this problem. Is $\nabla^2_{xx} L(x^*, \lambda^*)$ positive definite?

(b) [4 MARKS] Identify the critical cone at the point (x^*, λ^*).

(c) [6 MARKS] State a second-order sufficiency condition for x^* to be a local minimum of a NLP. Show that this condition holds here.

(d) [6 MARKS]Verify that constraint qualifications (CQ) hold at this point. Can you state whether this point is a local minimum of $f(x)$?

End of Examination

USEFUL FORMULAE

Result 1 *Taylor series expansion:*

$$f(x^* + y) = f(x^*) + \nabla f(x^*)^T y + \frac{1}{2} y^T \nabla^2 f(x^*) y + o(\|y\|^2)$$

Result 2 *Given a square matrix B, with full rank, the eigenvalues $\lambda_1, \ldots, \lambda_n$ and respective (normalised) eigenvectors $v_1, \ldots, v_n$ satisfy $Bv_i = \lambda_i v_i$, and $\{v_i, i = 1, \ldots, n\}$ are orthonormal (orthogonal and with unit norm). This set spans $\mathbb{R}^n$ and can be used as a basis.*

Result 3

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

SOLUTION TO PROBLEM 1:

[4 MARKS] Both methods search for the solution in an interval $[a, b]$ using two initial points $a < p < q < b$ and adding, at each iteration, only one more point to evaluate the function. From the f-calculations at the points, if $f(p) < f(q)$ then the subinterval $(q, b]$ can be discarded and the solution now lies in an interval of reduced size $q - a$. Otherwise, if $f(p) > f(q)$ then the discarded subinterval is $[a, p)$. Fibonacci chooses the points at each iteration so that the length of the discarded subinterval is independent of whether $f(p)$ is smaller or larger than $f(q)$, that is, $q - a = b - p$. This implies that consecutive iterations reduce the length of the subinterval by a fraction F_{k-1}/F_k where F_n is the n-th Fibonacci number. The Golden search algorithm replaces the fraction F_{k-1}/F_k when calculating the new point, with its limit value $\gamma = \lim_{n\to\infty} F_{k-1}/F_k$. Because it no longer balances the points, the length of the discarded intervals is no longer independent of the values of $f(p), f(q)$, and it is in general smaller than the fraction $1 - F_{k-1}/F_k$.

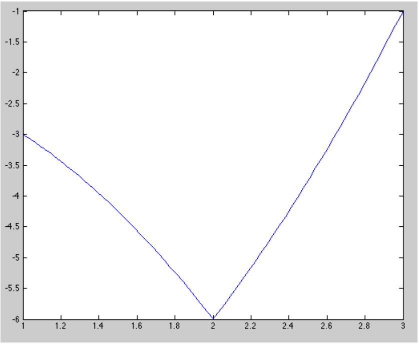
The number of iterations of the Fibonacci search will be smaller than the Golden Section, to achieve a precision ϵ, because the size of the reduced subintervals at each iteration is minimised by Fibonacci. However, the execution time of each iteration of the Fibonacci algorithm is larger than each iteration of the Golden search, because the latter uses a constant, instead of the ratio F_{k-1}/F_k. Even if these ratios were produced in advance and stored in a table to speed up execution time, the Fibonacci search method still requires initial computation of the number of iterations n required to achieve the required precision, using the smallest n such that $F_n \geq 2\epsilon/(b - a)$. It should be noted however, that Golden section requires a Boolean operation at each iteration for the stopping criterion.

SOLUTION TO PROBLEM 2:

(a) [2 MARKS] $f'(x) = -2x$, and $f''(x) = -2$ for $x < 2$. Thus $x^1 = 1.5 - (-2(1.5)/(-2)) = 0$. Because $f'(0) = 0$, then the algorithm will not move from this value, and so $\bar{x} = 0$ is the limit point.

(b) [2 MARKS] $f'(x) = 2x$, and $f''(x) = 2$ for $x > 2$. Thus $x^1 = 3 - (2(3)/2) = 0$. Because $f'(0) = 0$, then $\bar{x} = 0$ is the limit point.

(c) [4 MARKS]



The actual minimum occurs at $x^* = 2$, which is a point of discontinuity of the derivative of the function, so Newton's method is not guaranteed to converge to this solution. For $x \leq 2$ the function is *concave* and has a local maximum at $\bar{x} = 0$. Because it is a perfect quadratic, when $x^0 < 2$ Newton's method finds the *maximum* of $-x^2 - 2$ in one iteration, staring in this region. For $x > 2$ the function is a perfect quadratic and *convex*, so when $x^0 > 2$ Newton's method finds the minimum of $x^2 - 10$, which is also $\bar{x} = 0$, in one iteration. At that point, the function value corresponds to a local maximum, so the algorithm does not move.

SOLUTION TO PROBLEM 3:

The function is $f(x) = \frac{4}{3}x_1^3 - 2x_1x_2 + \frac{1}{2}x_2^2$.

(a) [2 MARKS] The gradient of f is:

$$\nabla f(x) = \begin{bmatrix} 4x_1^2 - 2x_2 \\ -2x_1 + x_2 \end{bmatrix}$$

Stationary points satisfy $4x_1^2 - 2x_2 = 0$ and $-2x_1 + x_2 = 0$, so $2x_1^2 - x_2 = 0$ and $x_2 = 2x_1$. Verification: $2(1) - 2 = 0, 2 = 2(1)$.

(b) [6 MARKS] The Hessian of the function is:

$$\nabla^2 f(x) = \begin{bmatrix} 8x_1 & -2 \\ -2 & 1 \end{bmatrix}$$

so we need to verify that this matrix at the point $x^* = (1,2)^T$ is positive definite. The eigenvalues of the matrix:

$$\nabla^2 f((1,2)^T) = \begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix}$$

satisfy $(8-\lambda)(1-\lambda) - 4 = 0$, or $\lambda^2 - 9\lambda + 4 = 0$, which yields $\lambda = \frac{9 \pm \sqrt{65}}{2}$, which are both positive, thus the Hessian at this point is positive definite and $(1,2)^T$ is a local minimum. It *cannot* be a global minimum, since the function has no global minimum: to show this, consider for example, as $x_1 \to -\infty$, the function value also decreases to $-\infty$.

SOLUTION TO PROBLEM 4:

(a) [4 MARKS] The gradient of $f(x)$ is

$$\nabla f(x) = \begin{bmatrix} 2x_1 - x_2 - 3 \\ 2x_2 - x_1 + 3 \end{bmatrix}.$$

Setting $\nabla f(x) = 0$ yields $x_2 = 2x_1 - 3$, so that $2(2x_1 - 3) - x_1 + 3 = 0 \Rightarrow 4x_1 - 6 - x_1 + 3 = 0 \Rightarrow x_1 = 1$, and $x_2 = 2(1) - 3 = -1$. The only stationary point is $x^* = (1,-1)^T$. The Hessian matrix at this point is:

$$\nabla^2 f(x^*) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which is positive definite: eigenvalues satisfy $(2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 2 \pm 1 = 1, 3$ both positive. Thus this is the minimum.

(b) [8 MARKS] Steepest descent direction is negative of gradient, so it is $-\nabla f(1,1) = (2,-4)^T$. To obtain x^1 the method uses $x^1 = (1,1)^T + t^*(2,-4)^T$, where $t^* = \arg\min_{t \geq 0} q(t)$, and

$$
\begin{aligned}
q(t) &= f(1+2t, 1-4t) = (1+2t)^2 + (1-4t)^2 - (1+2t)(1-4t) - 3(1+2t) + 3(1-4t) + 3 \\
&= 4 - 20t + 28t^2.
\end{aligned}
$$

Setting $q'(t) = 0$ yields $2(28)t - 20 = 0 \Rightarrow t = 10/28 = 5/14$, therefore

$$
x_1^1 = 1 + 2\left(\frac{5}{14}\right) = \frac{12}{7}, \qquad x_2^1 = 1 - 4\left(\frac{5}{14}\right) = -\frac{3}{7}.
$$

(c) [8 MARKS] Newton direction is $d = -[\nabla^2 f(1,1)]^{-1}\nabla f(1,1)$. The inverse matrix is:

$$
[\nabla^2 f(1,1)]^{-1} = \frac{1}{2^2 - (-1)^2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
$$

so that

$$
d = -\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}
$$

Descent direction d is such that $\nabla f(1,1)^T d < 0$, Verification:

$$
(-2,4)\begin{pmatrix} 0 \\ -2 \end{pmatrix} = -8 < 0.
$$

Therefore $x_1^1 = 1 + 0 = 1, \qquad x_2^1 = 1 - 2 = -1$, which is the solution: the method converges in one iteration because the function $f(x)$ is quadratic.

SOLUTION TO PROBLEM 5:

(a) [4 MARKS] A stationary point for this problem is feasible, so that $h(x^*) = 0$. The feasible directions $\mathcal{C}(x^*)$ are such that for small t, $c^T(x^* + td) + b = 0$, that is, $t(c^T d) = 0$. Identifying now $\nabla h(X^*) = c$, we obtain the result.

(b) [12 MARKS] The definition of a local minimum of this problem is a point x^* such that there exists a neighbourhood of x^* such that for all feasible x in the neighbourhood $f(x) > f(x^*)$. In this case, because h is affine, this means that $f(x^* + td) > f(x^*), d \in \mathcal{C}(x^*) = \{d \colon \nabla h(x^*)^T d = 0\}$. Use now Taylor approximation around x^* *along the feasible directions* to obtain:

$$
f(x^* + td) = f(x^*) + t\nabla f(x^*)^T d + \frac{t^2}{2} d^T \nabla^2 f(x^*) d + o(\|t\|^2),
$$

and use that x^* is stationary, so it satisfies the first order condition $\nabla_x L(x^* . \eta^*) = 0$, or $\nabla f(x^*)^T d = -\nabla h(x^*)^T d = 0$, because $d \in \mathcal{C}(x^*)$. Express now the vector $d \in \mathbb{R}^n$ as a linear combination of the eigenvectors of $\nabla^2 f(x^*)$, that is, $d = \alpha_1 v_1 + \ldots + \alpha_n v_n$ for real numbers $\alpha_i, i = 1, \ldots, n$. By assumption, $v_1^T d = 0$, because v_1 is co-linear with c. Therefore $\alpha_1 = 0$. To end the proof, we use $\nabla^2 f(x^*) v_i = \lambda_i v_i$, and the fact that the v_i's are orthogonal, which implies

$$f(x^* + td) - f(x^*) = \frac{t^2}{2} d^T \left(\sum_{i=2}^{n} \alpha_i \lambda_i v_i \right) + o(\|t\|^2).$$

For all non-zero $d \in \mathcal{C}(x^*)$, $\sum_{i=2}^{n} \lambda_i \alpha_i^2 \|v_i\|^2 > 0$, which implies that there exists a sufficiently small t such that $f(x^* + td) - f(x^*) > 0$, as required.

SOLUTION TO PROBLEM 6:

Call x_1 the hours of labour from permanent personnel, x_2 the hours of labour hired externally and x_3 the units of capital per week. The revenue is $270(x_1 + x_2)^{1/2} x_3^{1/3}$. The cost is $10x_1 + 25x_2 + 5x_3$. Therefore the problem is:

$$\begin{aligned} \min f(x) &= 10x_1 + 25x_2 + 5x_3 - 270(x_1 + x_2)^{1/2} x_3^{1/3} \\ \text{s.t. } \ & g_i(x) = -x_i \leq 0, \quad i = 1, 2, 3, \\ & g_4(x) = x_1 - 160 \leq 0. \end{aligned}$$

The Lagrangian for this problem is:

$$L(x, \lambda) = 10x_1 + 25x_2 + 5x_3 - 270(x_1 + x_2)^{1/2} x_3^{1/3} - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 + \lambda_4 (x_1 - 160).$$

Because the constraints here are affine, KKT holds at the optimal value. At the minimum x^*, the non-negativity constraints are **not** active: $g_i(x^*) < 0, i = 1, 2, 3$ thus from KKTb, $\lambda_i^* = 0, i = 1, 2, 3$. The KKTa conditions for stationarity therefore imply that at the minimum (x^*, λ^*),

$$\nabla_x L(x^*, \lambda^*) = \begin{pmatrix} 10 - \frac{270 x_3^{1/3}}{2\sqrt{x_1 + x_2}} + \lambda_4^* \\ 25 - \frac{270 x_3^{1/3}}{2\sqrt{x_1 + x_2}} \\ 5 - \frac{270\sqrt{x_1 + x_2}}{3 x_3^{2/3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows from the first and second equations that, necessarily, $\lambda_4^* = 15$ at the minimum. From KKTb, because $\lambda_4^* g_4(x^*) = 0$, we must have $x_1^* = 160$ as indicated. Using the sensitivity Theorem 7 in Lecture Notes, the optimal cost will *decrease* by $\lambda_4^* = \$15$ for every hour on top of the 160 that can be made available at a cost of \$10. Clearly this will yield a net profit of \$5 per extra hour of labour, so we would recommend the company to increase their permanent personnel instead of buying labour externally.

SOLUTION TO PROBLEM 7:

(a) [4 MARKS]

$$L(x,\lambda) = 3x_1^2 + 4x_1x_2 + \lambda(x_1 + 2x_2 - 3) \qquad \nabla L(x) = \begin{pmatrix} 6x_1 + 4x_2 + \lambda \\ 4x_1 + 2\lambda \end{pmatrix}$$

KKTa: $5(-3) + 4(3) + \lambda^* = 0 \Rightarrow \lambda^* = 6$, and also $4(-3) + 2(6) = 0$.

KKTb: $\lambda^* = 6 > 0$, and $g(x^*) = (-3) + 2(3) - 3 = 0$, so $\lambda^* g(x^*) = 0$.

KKTc: No equality constraints.

The Hessian at this point is:

$$\nabla^2_{xx} L(x^*, \lambda^*) = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix},$$

with eigenvalues satisfying $-\lambda(6-\lambda) = 16$, so $\lambda^2 - 6\lambda + 9 = 25 \Rightarrow \lambda = 3 \pm 5 = \{-2, 8\}$. Therefore the matrix is NOT positive definite.

(b) [4 MARKS] At this point there is one *active* constraint with $\lambda^* > 0$. we have $\nabla g(x) = (1,2)^T$. The critical cone at the point (x^*, λ^*) is

$$\begin{aligned} \mathcal{C}(x^*) &= \{d \in \mathbb{R}^2 \colon \nabla g(x^*)^T d = 0\} = \left\{ d\colon (1,2) \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0 \right\} \\ &= \{d\colon d_1 = -2d_2, d_1 \in \mathbb{R}\} \end{aligned}$$

(c) [6 MARKS] For $d \in \mathcal{C}(x^*)$, $d^T \nabla^2_{xx} L(x,\lambda) d > 0$. This condition holds here:

$$(-2d_2, d_2) \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} -2d_2 \\ d_2 \end{pmatrix} = (-2d_2, d_2) \begin{pmatrix} -8d_2 \\ -8d_2 \end{pmatrix} = 16d_2^2 - 8d_2^2 = 8d_2^2 > 0.$$

(d) [6 MARKS] g is the only constraint and it is active: $g(x^*) = -3 + 2(3) - 3 = 0$. We seek d such that it is a descent direction for g at x^*. Since $\nabla g(x^*) = (1,2)^T$, choosing for example $d = (-1,-1)^T$ we have $\nabla g(x^*)^T d = -1 - 2 = -3 < 0$, so CQ holds. From Theorem 10 in Lecture Notes, CQ at x^* plus the second order condition imply that x^* is indeed a local minimum for this problem.