620-361 Operations Research Techniques and Algorithms

Assignment 1 Solutions

1. (9 marks) **Consider the function**

$$f(x) = x \ln x - x + 5.$$

It is known that there is a minimum of f in the interval $[0.5, 1.2]$. Apply the

- **Fibonacci search;**
- **Golden section search;**
- **False position method;**
- **Newton's method**

to find an estimate for this minimum. Use at most 4 calculations of f or any of its derivatives for each method. Which method produces the most accurate estimate?

- **Fibonacci search**. As $F_4 = 5$, we have $p = 1.2 - \frac{3}{5}0.7 = 0.78, q = 0.5 + \frac{3}{5}0.7 = 0.92$. Calculating f gives us

$$f(0.78) = 4.026 > f(0.92) = 4.003$$

so we set $a = 0.78$, $p = 0.92$, $q = 0.78 + \frac{2}{3}(1.2 - 0.78) = 1.06$. Then

$$f(1.06) = 4.00176 < f(0.92)$$

so we set $a = 0.92$, $p = 1.06$. We set q to be near p: $q = 1.061$. We find

$$f(1.061) = 4.00182 > f(1.06)$$

so our final interval is $[0.92, 1.061]$, and our final estimate is 0.9905.

- **Golden section search**. We set $p = 1.2 - 0.7\gamma = 0.767$, $q = 0.5 + 0.7\gamma = 0.933$. Calculating f gives us

$$f(0.767) = 4.029 > f(0.933) = 4.002$$

so we set $a = 0.767$, $p = 0.933$, $q = 0.767 + (1.2 - 0.767)\gamma = 1.035$. Then

$$f(1.035) = 4.001 < f(0.933)$$

so we set $a = 0.933$, $p = 1.035$, $q = 0.933 + (1.2 - 0.933)\gamma = 1.098$. We find

$$f(1.098) = 4.005 > f(1.035)$$

so our final interval is $[0.933, 1.098]$, and our final estimate is 1.0152.

- **False position search**. Differentiating f gives us

$$f'(x) = \ln x + x\frac{1}{x} - 1 = \ln x.$$

Calculating f' at the endpoints gives

$$f'(0.5) = -0.693, f'(1.2) = 0.182$$

so $p = 0.5 + \frac{(1.2-0.5)f'(0.5)}{f'(0.5)-f'(1.2)} = 1.054$. Then

$$f'(1.054) = 0.053 > 0$$

so we set $b = 1.054$, $p = 0.5 + \frac{(1.054-0.5)f'(0.5)}{f'(0.5)-f'(1.054)} = 1.015$. Repeating gives

$$f'(1.015) = 0.015 > 0$$

so our final estimate is $p = 0.5 + \frac{(1.015-0.5)f'(0.5)}{f'(0.5)-f'(1.015)} = 1.0042$.

- **Newton's method**. We have $f''(x) = \frac{1}{x}$. We take the midpoint of our original interval, 0.85, as our beginning estimate a. Then

$$f'(0.85) = -0.163, f''(0.85) = 1.176$$

and $p = 0.85 - \frac{f'(0.85)}{f''(0.85)} = 0.988$. We set $a = p$ and calculate

$$f'(0.988) = -0.012, f''(0.988) = 1.012$$

and so our final estimate is $0.988 - \frac{f'(0.988)}{f''(0.988)} = 0.9999$.

The idea of a 'most accurate' estimate is slightly ambiguous; you may look at the value of the function at the estimate, or for the section searches you may choose to look at the length of the final interval rather than the final point estimate. However, we choose to look at the distance our estimate is from the true minimum. The true minimum occurs when $f'(x) = \ln x = 0$, i.e. $x = 1$. Clearly, Newton's method is the most accurate, followed by the false position method, Fibonacci search and finally golden section search.

2. (6 marks) **Prove that if you have at least 2 calculations available, Fibonacci search will always result in a strictly smaller interval than golden section search for the same number of calculations.**

Assume without loss of generality that we start with an interval of length 1. After n calculations (where $n \geq 1$), Fibonacci search results in an interval of length $\frac{1}{F_n}$, whereas golden section search results in an interval of length γ^{n-1}. So we merely need to prove that $F_n > \left(\frac{1}{\gamma}\right)^{n-1}$ for $n \geq 2$. To do this we use induction. It is easily seen from calculation that $F_2 > \frac{1}{\gamma}$ and $F_3 > \left(\frac{1}{\gamma}\right)^2$. Now assume that the relation holds up to some n. Then

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &> \left(\frac{1}{\gamma}\right)^{n-1} + \left(\frac{1}{\gamma}\right)^{n-2} \\ &= \left(\frac{1}{\gamma}\right)^{n} (\gamma + \gamma^2) \\ &= \left(\frac{1}{\gamma}\right)^{n}. \end{aligned}$$

By induction, the relation holds for all $n \geq 2$.

3. (5 marks + 5 bonus marks) **Prove that if the function $f(x)$ is bounded below on the interval $[0, \infty)$, there always exists a point which satisfies both the Armijo-Goldstein and Wolff conditions at the same time.**

   If $f'(0) \geq 0$, then $x = 0$ is a point which satisfies both the Amrijo-Goldstein and Wolff conditions. So we can assume that $f'(0) < 0$.

   Assume that no point satisfies the Wolff condition. Then for all $x \geq 0$, $f'(x) < \mu f'(0) < 0$, and hence $f(x) \leq f(0) + \mu f'(0)x$. This means that f is unbounded below, a contradiction. Therefore there must exist a point which satisfies the Wolff condition.

   Now let x^* be the smallest $x \in [0, \infty)$ which satisfies the Wolff condition. By definition, all $x < x^*$ do not satisfy the Wolff condition. In particular, this means that $f'(x^*) = \mu f'(0)$. So for all $x \leq x^*$,

   $$f'(x) \leq \mu f'(0) \leq \sigma f'(0)$$

   and hence

   $$f(x) \leq f(0) + \sigma f'(0)x.$$

   This means that x^* satisfies both the Armijo-Goldstein condition and the Wolff condition.