620-361 Operations Research Techniques and Algorithms

Assignment 3 Solutions

1. [8 marks] **Consider the non-linear program**

$$\begin{array}{ll} \textbf{min} & -x_1 - x_2 \\ \textbf{s.t.} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1^2 + \frac{1}{2}x_2^2 \leq 1. \end{array}$$

(a) Write down the Lagrangian of this program.

(b) Write down the KKT conditions for this program.

(c) Solve the KKT conditions.

(d) Show that a constraint qualification holds at your KKT point.

(e) Apply a sufficient condition to confirm that your KKT point is a minimum.

(a)

$$L(x, \lambda) = -x_1 - x_2 + \lambda_1(-x_1) + \lambda_2(-x_2) + \lambda_3(x_1^2 + \frac{1}{2}x_2^2 - 1).$$

(b) KKTa:

$$\nabla L(x, \lambda) = \begin{bmatrix} -1 - \lambda_1 + 2\lambda_3 x_1 \\ -1 - \lambda_2 + \lambda_3 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

KKTb:

$$-x_1 \leq 0, \lambda_1 \geq 0, \lambda_1(-x_1) = 0$$

$$-x_2 \leq 0, \lambda_2 \geq 0, \lambda_2(-x_2) = 0$$

$$x_1^2 + \frac{1}{2}x_2^2 - 1 \leq 0, \lambda_3 \geq 0, \lambda_3(x_1^2 + \frac{1}{2}x_2^2 - 1) = .0$$

(c) We know that $2\lambda_3 x_1 = 1 + \lambda_1 > 0$, so neither x_1 nor λ_3 can be 0. Hence

$$\lambda_1 = 0.$$

We also know that $x_1^2 + \frac{1}{2}x_2^2 - 1 = 0$. Similarly, $\lambda_3 x_2 = 1 + \lambda_2 > 0$, so x_2 cannot be 0 and

$$\lambda_2 = 0.$$

Now $2\lambda_3 x_1 = 1 = \lambda_3 x_2$, and we know that $\lambda_3 \neq 0$, so $2x_1 = x_2$. Therefore $x_1^2 + 2x_1^2 = 1$, which (coupled with $x_1 \geq 0$) implies that

$$x_1 = \frac{1}{\sqrt{3}}.$$

Back substitution then gives us

$$x_2 = \frac{2}{\sqrt{3}}, \lambda_3 = \frac{\sqrt{3}}{2}.$$

(d) The only active constraint at the KKT point is the third constraint. Therefore the only active gradient is

$$\nabla g_3(x) = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}.$$

At the KKT point this is

$$\nabla g_3(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}) = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{bmatrix}.$$

Taking $d = (-1, -1)^T$ clearly satisfies the Mangasarian-Fromovitz constraint qualification.

(e) The Hessian of the Lagrangian is

$$\nabla^2 L(x, \lambda) = \begin{bmatrix} 2\lambda_3 & 0 \\ 0 & \lambda_3 \end{bmatrix}.$$

At the KKT point this is

$$\nabla^2 L(x, \lambda) = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

This is positive definite, and so clearly positive definite on the critical cone. Therefore the KKT point is a minimum.

2. [7 marks] **Consider the non-linear program in question 1.**

(a) Write down the l_2 penalty function for this program with penalty parameter α.

(b) Find the gradient of the penalty function when $\alpha = \frac{1}{4}$.

(c) Find a minimum of the penalty function when $\alpha = \frac{1}{4}$ and $x_1 > 0$, $x_2 > 0$, and $x_1^2 + \frac{1}{2}x_2^2 > 1$.

(d) The above penalty function has a minimum at an infeasible point. Do you expect the penalty function to have an infeasible minimum for $\alpha = 1000$? Why or why not?

(a)

$$P_\alpha(x) = -x_1 - x_2 + \frac{\alpha}{2}\left((-x_1)_+^2 + (-x_2)_+^2 + (x_1^2 + \frac{1}{2}x_2^2 - 1)_+^2\right).$$

(b)

$$\nabla P_{1/4}(x) = \begin{bmatrix} -1 - \frac{1}{4}(-x_1)_+ + \frac{1}{2}x_1(x_1^2 + \frac{1}{2}x_2^2 - 1)_+ \\ -1 - \frac{1}{4}(-x_2)_+ + \frac{1}{4}x_2(x_1^2 + \frac{1}{2}x_2^2 - 1)_+ \end{bmatrix}.$$

(c) In the specified region,

$$\nabla P_{1/4}(x) = \begin{bmatrix} -1 + \frac{1}{2}x_1(x_1^2 + \frac{1}{2}x_2^2 - 1) \\ -1 + \frac{1}{4}x_2(x_1^2 + \frac{1}{2}x_2^2 - 1) \end{bmatrix}.$$

This means that

$$\frac{1}{2}x_1(x_1^2 + \frac{1}{2}x_2^2 - 1) = 1 = \frac{1}{4}x_2(x_1^2 + \frac{1}{2}x_2^2 - 1)$$

which implies

$$x_2 = 2x_1.$$

Then

$$-1 + \frac{1}{2}x_1(x_1^2 + 2x_1^2 - 1) = \frac{3}{2}x_1^3 - \frac{1}{2}x_1 - 1 = 0.$$

The only real solution of this cubic is $x_1 = 1$, which then gives $x_2 = 2$.

(d) The minimum will also be infeasible. This is because the penalty is quadratically small near the boundary of the constraint, and any decrease in the function will 'override' it for small violations.

3. [5 marks] **Show that any point on a line drawn between any two feasible points in a convex program is itself feasible. (In other words, that the feasible region is a convex set).**

If the points are x and y, then any point on the line between then can be expressed as $\alpha x + (1 - \alpha)y$ for some $\alpha \in [0, 1]$. Since $g_i(x)$ is convex,

$$g_i(\alpha x + (1 - \alpha)y) \le \alpha g_i(x) + (1 - \alpha)g_i(y) \le 0$$

since $\alpha \in [0, 1]$ and both x and y are feasible. Furthermore, since $h(x)$ is affine,

$$h(\alpha x + (1 - \alpha)y) = \alpha h(x) + (1 - \alpha)h(y) = 0.$$

Therefore any point on the line is feasible.