620-361 Operations Research Techniques and Algorithms

Semester 1 Exam — June, 2006

DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF MELBOURNE

Exam duration: three hours
Reading time: fifteen minutes
This paper has four (4) pages–including this page

Authorised materials:
Non-programmable calculators are permitted.

Instructions to invigilators:
The exam paper may be taken out of the examination room.

Instructions to students:
There are seven (7) questions. All questions may be attempted.
The number of marks for each question is indicated.
The total number of marks available is 100
Useful formulae are provided on page 4.

This paper is to be lodged with the Baillieu Library

PROBLEM 1 [12 marks]
Let $f \colon \mathbb{R} \to \mathbb{R}$ be a unimodal function. We wish to use a numerical line search method to estimate its minimum on the closed interval $[a, b]$, with a pre-specified error tolerance of $\epsilon = .01$.

(a) Explain why the number of iterations required for the Fibonacci search method is in general smaller than the number required for the Golden search method, when the two are applied to achieve the same tolerance ϵ.

(b) Explain why in general the Golden search method requires less CPU time than the Fibonacci search method, even if more iterations are required.

(c) Let $f(x) = x^2 - 4x + 7$, and $x^0 = 0$ an initial point. Perform **one** iteration of Newton's method to find $\min f(x)$. Evaluate the error in the estimate and explain your result.

PROBLEM 2 [8 marks]
Consider the function $f(x) = 3x_1^2 + 4x_1x_2 + 6x_1 + 12x_2 + 30$.

(a) Show that the point $x^* = (-3, 3)$ is a stationary point.

(b) What kind of stationary point (max, min, saddle) is x^*? Explain your answer.

PROBLEM 3 [12 marks]
Let $f \colon \mathbb{R}^n \to \mathbb{R}, f \in C^2$. Consider a stationary point $x^* \in \mathbb{R}^n$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\nabla^2 f(x^*)$. Suppose that $\lambda_1 < 0$ and $\lambda_2 > 0$. Show that x^* is neither a local minimum nor a local maximum.

PROBLEM 4 [20 marks]
Consider the unconstrained nonlinear program $\min_{x \in \mathbb{R}^2} f(x) = \frac{4}{3}x_1^3 - x_1x_2^2 - 8x_2 + 3x_2^2$.

(a) Show that $x = (1, 2)^T$ is a stationary point of f.

(b) Does the second-order sufficiency condition hold for the stationary point $x = (1, 2)^T$? Is $x = (1, 2)^T$ a local minimum of f? Briefly justify your answer.

(c) Find the direction of steepest descent for f at the point $x^0 = (2, 0)^T$. The steepest descent algorithm uses a stepsize t^* to calculate x^1, of the form $t^* = \arg\min_{t \geq 0} q(t)$. Write down the function $q(t)$ for this problem (note that you only have to write down the function $q(t)$, you are not required to find its minimum, or to simplify your expression for $q(t)$).

(d) Find the Newton direction for f at $x^0 = (2, 0)^T$. Is the Newton direction a descent direction? Justify your answer.

(e) Show that a step of length $t_0 = 1$ along the Newton direction d at x^0 satisfies the Armijo-Goldstein condition, with the linesearch parameter $\sigma = \frac{1}{9}$.

PROBLEM 5 [14 marks]
Consider the problem $\min f(x) = x_1^2 + e^{-x_2^2}$.

(a) Show that $x = (0,0)^T$ is a stationary point of the unconstrained problem.

(b) Consider now the problem

$$\begin{aligned} \min f(x) &= x_1^2 + e^{-x_2^2} \\ \text{s.t} \quad & x_1 + x_2 = 1. \end{aligned}$$

Show that a KKT point must satisfy $\eta^* = -2x_1^*, x_1^* = -x_2^* e^{-x_2^{*2}}$.

(c) Using the second order conditions, determine whether the stationary point is a minimum or a maximum. You may use the result that the only solution to the equation $e^{-y^2} = 1 - 1/y$ is given by $y = 1.2583$.

(d) Estimate the approximate change in the optimal cost $f(x^*)$ if the constraint is changed by 10% to $x_1 + x_2 = 1.1$.

PROBLEM 6 [18 marks]
Consider the non linear utility maximisation problem:

$$\begin{aligned} \max f(x) &= \ln x_1 + 2\ln x_2 + 3\ln x_3 \\ \text{s.t.} & \\ x_1 + x_2 + x_3 &= 300 \\ x_1 &\geq 75 \end{aligned}$$

(a) Reformulate the problem as a minimisation problem and write down the Lagrangian. Show that $(x^*, \lambda^*, \eta^*) = (75, 90, 135, \frac{6}{675}, \frac{1}{45})^T$ is a KKT point for this problem.

(b) Show that there is no KKT point when the inequality constraint is inactive.

(c) Identify the critical cone at the point (x^*, λ^*).

(d) State a second-order sufficiency condition for x^* to be a local minimum of a NLP. Show that this condition holds here. Can you state whether this point is a global maximum of f(x)?

PROBLEM 7 [16 marks]
Consider the constrained nonlinear program

$$\min_{x\in\mathbb{R}^2} f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 \quad \text{subject to} \quad x_1, x_2 \leq 0.$$

(a) Write down the l_2-penalty function $P_k(x)$ with penalty parameter k, and explain in general terms, how the ℓ_2-penalty method approximates a solution to a constrained nonlinear program.

(b) Simplify $P_k(x)$ when $x_1 > 0$ and $x_2 < 0$. Write down $\nabla P_k(x)$ when $x_1 > 0$ and $x_2 < 0$.

(c) Find a stationary point $x^k = (x_1^k, x_2^k)$ for $P_k(x)$ such that $x_1^k > 0$ and $x_2^k < 0$. Write down the limit $x^* = \lim_{k\to\infty} x^k$.

(d) Write down an estimate λ^k of the optimal multiplier vector, and find the limit $\lambda^* = \lim_{k\to\infty} \lambda^k$.

End of Examination

USEFUL FORMULAE

Result 1 *Taylor series expansion:*

$$f(x^* + y) = f(x^*) + \nabla f(x^*)^T y + \frac{1}{2} y^T \nabla^2 f(x^*) y + o(\|y\|^2)$$

Result 2 *Given a square matrix B, with full rank, then the eigenvalues $\lambda_1, \ldots, \lambda_n$ and respective (normalised) eigenvectors $v_1, \ldots, v_n$ satisfy $Bv_i = \lambda_i v_i$, and $\{v_i, i = 1, \ldots, n\}$ are orthonormal (orthoogonal and with unit norm).*

KKT(a) $\nabla_x L(x^*, \lambda^*, \eta^*) = 0$

KKT(b) $g_i(x^*) \leq 0, \lambda_i^* \geq 0, \lambda_i^* g_i(x^*) = 0$

KKT(c) $h_j(x^*) = 0$

SOLUTION TO PROBLEM 1:

(a) [4 MARKS] The Fibonacci search re-uses f calculations so it is designed to be efficient in the number of calculations (iterations). To do so, the algorithm is *unbiased*, that is, it requires the same number of iterations to achieve the given tolerance for any unimodal function. The Golden ratio search is not optimal and the search is biased.

(b) [4 MARKS] The Fibonacci search method requires calculation of the ratios F_{n+1}/F_n of consecutive Fibonacci numbers at each iteration, and of course it also needs to compute $(F_i, i = 1 \dots N)$. The Golden search method uses a constant number γ at each iteration, and only requires a computation of the current interval length to stop the algorithm. These computations are usually much faster, compensating for the CPU time.

(c) [4 MARKS] $f'(x) = 2x - 4$, $f''(x) = 2$. The iteration yields:

$$x^1 = x^0 - \frac{f'(x^0)}{f''(x^0)} = 0 - \frac{-4}{2} = 2.$$

To estimate the actual error, we evaluate the true optimum: from the first order conditions, $f'(x^*) = 0$ implies $x^* = 2$, and this is a minimum because f is convex: $f'' > 0$. Therefore the error after one iteration is zero: the algorithm finds the minimum in just one step because the function is *quadratic*.

SOLUTION TO PROBLEM 2:

(a) [2 MARKS] At this point we must have $\nabla f(x^*) = 0$, or

$$\begin{aligned} 6x_1 + 4x_2 + 6 &= 0, & 4x_1 + 12 &= 0, \\ 6(-3) + 4(3) + 6 &= 0, & 4(-3) + 12 &= 0. \end{aligned}$$

(b) [6 MARKS] Hessian of function:

$$\nabla f(x) = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$$

is independent of x. We must verify if positive (negative) definite for minumum (maximum), etc. Eigenvalues satisfy

$$\begin{aligned} -\lambda(6 - \lambda) &= 16 \\ \lambda^2 - 6\lambda + 9 &= 16 + 9 \\ (\lambda - 3)^2 &= 25 \\ \lambda &= 3 \pm 5. \end{aligned}$$

So $\lambda_1 = -2, \lambda_2 = 8$. If all eigenvalues of a symmetric matrix are positive (negative) then the matrix is positive (negative) definite. In this case there is a negative and a positive eigenvalue, so x^* is a saddle point.

SOLUTION TO PROBLEM 3:

Let v_1, v_2 be the eigenvectors corresponding to λ_1, λ_2 respectively. Use Taylor's expansion, for small t, to obtain:

$$\begin{aligned} f(x^* + tv_i) &= f(x^*) + \nabla f(x^*)^T(tv_i) + \frac{1}{2}(tv_i^T)\nabla^2 f(x^*)(tv_i) + o(t^2) \\ &= f(x^*) + \frac{t^2}{2}\lambda_i v_i^T v_i + o(t^2) \\ &= f(x^*) + \frac{t^2}{2}\,\lambda_i \end{aligned}$$

because $v_i^T v_i = \|v_i\|^2 = 1$.

Therefore $\lambda_1 < 0$ implies x^* is a local MAXIMUM in this direction, and $\lambda_2 > 0$ implies x^* is a local MINIMUM in this direction.

SOLUTION TO PROBLEM 4:

(a) [2 MARKS] First order condition $\nabla f(x^*) = 0$:

$$\nabla f(x) = \begin{pmatrix} 4x_1^2 - x_2^2 \\ -2x_1x_2 - 8 + 6x_2 \end{pmatrix}, \quad \text{so:} \quad \nabla f(1,2) = \begin{pmatrix} 4-4 \\ -4-8+12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b) [4 MARKS] Hessian:

$$\nabla^2 f(x) = \begin{pmatrix} 8x_1 & -2x_2 \\ -2x_2 & -2x_1+6 \end{pmatrix}, \quad \text{so:} \quad \nabla^2 f(1,2) = \begin{pmatrix} 8 & -4 \\ -4 & 4 \end{pmatrix}$$

which is positive definite: $\det(\nabla^2 f(1,2)) = 32 - (-16) = 48 > 0$. Alternatively, solve for the eigenvalues: $(8-\lambda)(4-\lambda) - 16 = 0 \Rightarrow \lambda^2 - 12\lambda + 16 = 0 \Rightarrow \lambda = \frac{12 \pm \sqrt{80}}{2} > 0$ for both roots. Therefore the point $(1,2)^T$ is a local minimum.

(c) [6 MARKS] Steepest descent direction is negative of gradient, so it is $-\nabla f(2,0) = (-16,8)^T$. To obtain x^1 the method uses $x^1 = (2,0)^T + t^*(-16,8)^T$, where $t^* = \arg\min_{t\geq 0} q(t)$, and

$$\begin{aligned} q(t) &= f[(2,0) + t(-16,8)^T] = f(2-16t, 8t) = \frac{4}{3}(2-16t)^3 - (8t)^2(2-16t) + 64t + 3(-8t)^3 \\ &= \frac{4}{3}(2-16t)^3 + 1024t^3 + 64t^2 - 64t. \end{aligned}$$

(d) [4 MARKS] Newton direction is $-[\nabla^2 f(2,0)]^{-1}\nabla f(2,0)$, that is:

$$\begin{pmatrix} 16 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -16 \\ 8 \end{pmatrix} = \begin{pmatrix} 1/16 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} -16 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Descent direction d is such that $\nabla f(2,0)^T d < 0$, Verification:

$$\nabla f(2,0)^T \begin{pmatrix} -1 \\ 4 \end{pmatrix} = (16, -8) \begin{pmatrix} -1 \\ 4 \end{pmatrix} = -16 - 32 = -48 < 0.$$

(e) [4 MARKS] Use $t = 1$ and $d = (-1, 4)^T$. Armijo-Goldstein is satisfied if:

$$f(x^0 + td) \leq f(x^0) + t\sigma \nabla f(x^0)^T d.$$

Verification: $x^0 + td = (2, 0)^T + (1) \times (-1, 4)^T = (1, 4)^T$. $f(1, 4) = 4/3$, and

$$f(x^0) + t\sigma \nabla f(x^0)^T d = \frac{32}{3} + 1\left(\frac{1}{9}\right)(-48) = \frac{48}{9} > \frac{4}{3}.$$

SOLUTION TO PROBLEM 5:

(a) [2 MARKS] First order condition:

$$\nabla f(x) = \begin{pmatrix} 2x_1 \\ -2x_2 e^{-x_2^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is readily verified at $x = (0, 0)^T$.

(b) [2 MARKS] $L(x, \eta) = x_1^2 + e^{-x_2^2} + \eta(x_1 + x_2 - 1)$, so that KKT(a) requires:

$$\begin{aligned} 2x_1 + \eta &= 0 \\ -2x_2 e^{-x_2^2} + \eta &= 0 \end{aligned}$$

so that $x_1^* = -x_2^* e^{-x_2^{*2}}$, from the first equation, $\eta^* = -2x_1^*$.

(c) [6 MARKS] Use $h(x^*) = 0$ to get: $-x_2^* e^{-x_2^{*2}} + x_2^* = 1$, or $e^{-x_2^{*2}} = 1 - \frac{1}{x_2^*}$, so that $x_2^* = 1.2583$, and $h(x^*) = 0$ gives $x_1^* = 1 - x_2^* = -0.2583$. The Hessian of the Lagrangian is:

$$\nabla_{xx}^2 L(x, \eta) = \begin{pmatrix} 2 & 0 \\ 0 & (4x_2^2)e^{-x_2^2} - 2e^{-x_2^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & (4x_2^2 - 2)e^{-x_2^2} \end{pmatrix}$$

which is positive definite at x^*, because $4(1.2583)^2 > 2$ (diagonal matrix). So x^* is a local minimum.

(d) [6 MARKS] The change in constraint value is $\Delta = 0.1$, thus by the sensitivity Theorem, $\Delta f(x^*) \approx -\eta^*(0.1)$. $\eta^* = -2x_1^* = -2(-0.2583) = 0.5166$, so the estimated change is a **decrease** in cost of 0.05166.

SOLUTION TO PROBLEM 6:

(a) [4 MARKS] $\min -\ln x_1 - 2\ln x_2 - 3\ln x_3$, subject to $h(x) = x_1 + x_2 + x_3 - 300 = 0$ and $g(x) = 75 - x_1 \leq 0$. Lagrangian is:

$$L(x, \lambda, \eta) = -\ln x_1 - 2\ln x_2 - 3\ln x_3 + \lambda(75 - x_1) + \eta(x_1 + x_2 + x_3 - 300).$$

Stationary point must satisfy KKT(a):

$$\begin{aligned} -\frac{1}{x_1} + \eta - \lambda &= 0 \\ -\frac{2}{x_2} + \eta &= 0 \\ -\frac{3}{x_3} + \eta &= 0 \end{aligned}$$

Verification:

$$\begin{aligned} -\frac{1}{75} + \frac{1}{45} - \frac{6}{675} = \frac{-9 + 15 - 6}{675} &= 0 \\ -\frac{2}{90} + \frac{1}{45} &= 0 \\ -\frac{3}{135} + \frac{1}{45} = \frac{-3 + 3}{135} &= 0 \end{aligned}$$

$h(x^*) = 75 + 90 + 135 = 300$, and $x_1 = 75$, so the constraint g is **active**.

(b) [4 MARKS] In this case $\lambda = 0$, leading to:

$$\eta = \frac{1}{x_1} = \frac{2}{x_2} = \frac{3}{x_3}, \Rightarrow \frac{1}{\eta} + \frac{2}{\eta} + \frac{3}{\eta} = 300. \Rightarrow 6 = 300\eta, \Rightarrow \eta = 50,$$

but then $x_1 = 50, x_2 = 100, x_3 = 150$, which violates the inequality constraint: $x_1 < 75$.

(c) [4 MARKS] Critical cone:

$$\mathcal{C}(x^*, \eta^*) = \left\{ d \in \mathbb{R}^3 \colon \nabla h(x^*)^T d = 0, \nabla g(x^*)^T d = 0 \right\}$$

Use $\nabla h(x) = (1, 1, 1)^T, \nabla g(x) = (1, 0, 0)^T$ to get $d_1 + d_2 + d_3 = 0$ and $d_1 = 0$, so that $d_3 = -d_2$, $d_2 \in \mathbb{R}$.

(d) [6 MARKS] Hessian of Lagrangian is:

$$\nabla^2_{xx} L(x, \lambda, \eta) = \begin{pmatrix} \frac{1}{x_1^2} & 0 & 0 \\ 0 & \frac{2}{x_2^2} & 0 \\ 0 & 0 & \frac{3}{x_3^2} \end{pmatrix}$$

which at the point $x^* = (75, 90, 135)$ gives a positive definite matrix (diagonal with strictly positive entries). In particular, for $d \in \mathcal{C}(x^*)$,

$$(0, d_2, -d_2)\nabla^2_{xx} L(x, \lambda, \eta) d = (0, d_2, -d_2) \begin{pmatrix} 0 \\ \frac{d_2}{90^2} \\ -\frac{d_2}{135^2} \end{pmatrix} = \frac{d_2^2}{90^2} + \frac{d_2^2}{135^2} > 0.$$

We can state that this point is a global minimum of this problem because the function $-f(x)$ is convex, and both h and g are affine functions. Thus it is a global maximum of $f(x)$.

SOLUTION TO PROBLEM 7:

(a) [6 MARKS] Penalty function:

$$P_k(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 + \frac{k}{2}[(x_1)_+]^2 + \frac{k}{2}[(x_2)_+]^2$$

The penalty method adds a “cost” proportional to the amount by which each constraint $g_i(x) \le 0$ is violated. As such, as the penalty parameter k increases, the cost of violating the constraints increases. The method then finds the **unconstrained** optimal value: $x^k = \arg\min P_k(x)$. Because $P_k(x) = f(x)$ when x is feasible, then x^k will (1) be closer to the feasible region as k increases, and (2) approach the constrained optimum x^* of $f(x)$.

(b) [2 MARKS] Let $x_1 > 0$ and $x_2 < 0$. Then

$$P_k(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 + \frac{k}{2}x_1^2$$
$$\nabla P_k(x) = \begin{pmatrix} (1+k)x_1 - 1 \\ x_2 + 1 \end{pmatrix}$$

(c) [4 MARKS] Setting $\nabla P_k(x) = 0$ for $x_1 > 0$ and $x_2 < 0$ gives $x_1^k = 1/(1+k), x_2^k = -1$, which is stationary for $P_k(x)$ in the required region. $x_1^* = \lim_{k\to\infty} x_1^k = 0$, so the limit point is $x^* = (0, -1)^T$.

(d) [4 MARKS] For the region $x_1 > 0$ and $x_2 < 0$,

$$\lambda^k = \begin{pmatrix} k\left(\frac{1}{1+k}\right)_+ \\ k(-1)_+ \end{pmatrix} = \begin{pmatrix} \frac{k}{1+k} \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$