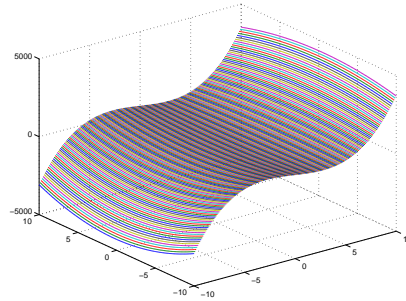


620-361 Operations Research Techniques and Algorithms  
Practice Class 2

1 Consider the unconstrained nonlinear program:

$$\min_{x \in \mathbb{R}^2} f(x) = \frac{4}{3}x_1^3 - x_1x_2^2 - 8x_2 + 3x_2^2.$$



(a) Show that  $x = (1, 2)^T$  is a stationary point of  $f$ .

First order condition  $\nabla f(x^*) = 0$ :

$$\nabla f(x) = \begin{pmatrix} 4x_1^2 - x_2^2 \\ -2x_1x_2 - 8 + 6x_2 \end{pmatrix}, \quad \text{so:} \quad \nabla f(1, 2) = \begin{pmatrix} 4 - 4 \\ -4 - 8 + 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore  $(1, 2)$  is a stationary point of  $f$ .

(b) Does the second-order sufficiency condition hold for the stationary point  $x = (1, 2)^T$ ? Is  $x = (1, 2)^T$  a local minimum of  $f$ ? Briefly justify your answer.

Hessian:

$$\nabla^2 f(x) = \begin{pmatrix} 8x_1 & -2x_2 \\ -2x_2 & -2x_1 + 6 \end{pmatrix}, \quad \text{so:} \quad \nabla^2 f(1, 2) = \begin{pmatrix} 8 & -4 \\ -4 & 4 \end{pmatrix}$$

which is positive definite:  $\det(\nabla^2 f(1, 2)) = 32 - (-16) = 48 > 0$ . Hence by Lemma 3,  $(1, 2)$  is a local minimum. Alternatively, solve for the eigenvalues:  $(8 - \lambda)(4 - \lambda) - 16 = 0 \Rightarrow \lambda^2 - 12\lambda + 16 = 0 \Rightarrow \lambda = \frac{12 \pm \sqrt{80}}{2} > 0$  for both roots. As all eigenvalues are positive, the point  $(1, 2)^T$  is a local minimum.

(c) Find the direction of steepest descent for  $f$  at the point  $x^0 = (2, 0)^T$ . The steepest descent algorithm uses a stepsize  $t^*$  to calculate  $x^1$ , of the form  $t^* = \arg \min_{t \geq 0} q(t)$ . Write down the function  $q(t)$  for this problem (note that you only have to write down the function  $q(t)$ , you are not required to find its minimum, or to simplify your expression for  $q(t)$ ).

### The steepest descent algorithm

To minimise a unimodal function  $f : \mathfrak{R}^N \rightarrow \mathfrak{R}$  to within tolerance  $\epsilon$ .

1. Select  $x^0 \in \mathfrak{R}^N$ .  
Set  $k = 0$ .
2. Calculate  $d^k = -\nabla f(x^k)$ .  
If  $\|d^k\| < \epsilon$  then stop.
3. Select step length  $t_k$  by solving the single-variable minimisation problem:  
 $\min q(t) = f(x^k + td^k)$ .
4. Set  $k = k + 1$ .  
Set  $x^{k+1} = x^k + t_k d^k$ .  
Return to step 2.

Steepest descent direction is negative of gradient, so it is  $-\nabla f(2, 0) = (-16, 8)^T$ . To obtain  $x^1$  the method uses  $x^1 = (2, 0)^T + t^*(-16, 8)^T$ , where  $t^* = \arg \min_{t \geq 0} q(t)$ , and

$$\begin{aligned} q(t) &= f[(2, 0) + t(-16, 8)^T] = f(2 - 16t, 8t) \\ &= \frac{4}{3}(2 - 16t)^3 - (8t)^2(2 - 16t) + 64t + 3(-8t)^3 \\ &= \frac{4}{3}(2 - 16t)^3 + 1024t^3 + 64t^2 - 64t. \end{aligned}$$

(d) Find the Newton direction for  $f$  at  $x^0 = (2, 0)^T$ . Is the Newton direction a descent direction? Justify your answer.

### Newton's Method

To minimise a unimodal function  $f : \mathfrak{R}^N \rightarrow \mathfrak{R}$  to within tolerance  $\epsilon$ .

1. Select  $x^0 \in \mathfrak{R}^N$ .  
Set  $k = 0$ .
2. If  $\|\nabla f(x^k)\| < \epsilon$  then stop.  
If  $\nabla^2 f(x^k)$  is positive definite, then  
Set  $d^k = -\nabla^2 f(x^k)^{-1} \nabla f(x^k)$ .  
Else, set  $d^k = -\nabla f(x^k)$

3. Select step length  $t_k$  either

- by solving the single-variable minimisation problem:  $\min q(t) = f(x^k + td^k)$ .
- by using our procedure for finding a step length that satisfies the Armijo-Goldstein and Wolff conditions.

4. Set  $k = k + 1$ .

Set  $x^{k+1} = x^k + t_k d^k$ .

Return to step 2.

Newton direction is  $-\left[\nabla^2 f(2, 0)\right]^{-1} \nabla f(2, 0)$ , that is:

$$\begin{pmatrix} 16 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -16 \\ 8 \end{pmatrix} = \begin{pmatrix} 1/16 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} -16 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Consider Lemma 6:

**Lemma 6**

If  $\nabla f(x^k) \neq 0$  and  $\nabla^2 f(x^k)$  is positive definite, then  $\nabla^2 f(x^k)$  is invertible and the Newton direction is a descent direction for  $f$  at  $x^k$ .

We can see that  $\nabla^2 f(x^k)$  is positive definite as it has only positive eigenvalues.

(e) Show that a step of length  $t_0 = 1$  along the Newton direction  $d$  at  $x^0$  satisfies the Armijo-Goldstein condition, with the linesearch parameter  $\sigma = \frac{1}{9}$ .

Use  $t = 1$  and  $d = (-1, 4)^T$ . Armijo-Goldstein is satisfied if:

$$f(x^0 + td) \leq f(x^0) + t\sigma \nabla f(x^0)^T d.$$

Verification:  $x^0 + td = (2, 0)^T + (1) \times (-1, 4)^T = (1, 4)^T$ .  $f(1, 4) = 4/3$ , and

$$f(x^0) + t\sigma \nabla f(x^0)^T d = \frac{32}{3} + 1 \left( \frac{1}{9} \right) (-48) = \frac{48}{9} > \frac{4}{3}.$$