620-361 Operations Research Techniques and Algorithms

Practice Class 3

1. Let $f\colon \Re^n \to \Re, f \in C^2$. Consider a stationary point $x^* \in \Re^n$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\nabla^2 f(x^*)$. Suppose that $\lambda_1 < 0$ and $\lambda_2 > 0$. Show that x^* is neither a local minimum nor a local maximum.

   Solution. Let v_1, v_2 be eigenvectors of norm 1 corresponding to λ_1, λ_2 respectively. Use Taylor's expansion, for small t, to obtain:

$$\begin{aligned} f(x^* + tv_i) &= f(x^*) + \nabla f(x^*)^T(tv_i) + \frac{1}{2}(tv_i^T)\nabla^2 f(x^*)(tv_i) + o(t^2) \\ &= f(x^*) + \frac{t^2}{2}\lambda_i v_i^T v_i + o(t^2) \\ &= f(x^*) + \frac{t^2}{2}\,\lambda_i + o(t^2) \end{aligned}$$

   because $v_i^T v_i = \|v_i\|^2 = 1$.

   For small enough t, the second term in this expression will dominate the third term. Therefore $\lambda_1 < 0$ implies x^* is a local maximum in this direction, and $\lambda_2 > 0$ implies x^* is a local minimum in this direction.

2. For each of the following matrices state whether it is positive definite, negative definite, or neither:

   Solution. A matrix, M, is positive definite if for all nonzero real-valued vectors x, $x^T Mx > 0$. For a symmetric matrix, this is equivalent to having all its eigenvalues greater than 0, or all its principal minors positive[1]. A matrix is positive semi-definite if $x^T Mx \geq 0$. Similarly, we have a negative definite matrix if $x^T Mx < 0$, and a negative semi-definite matrix if $x^T Mx \leq 0$.

   (a)

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

   The first leading principal minor is 1, which is > 0. The second leading principal minor is the matrix itself. The determinant of this

[1] The Sylvester criterion states that if all the determinant of the leading principal minors of M are positive, then M is positive definite. The leading principal minors are the upper left 1-by-1 corner of M; the upper left 2-by-2 corner of M; the upper left 3-by-3 corner of M; $\ldots$; M itself. In general, this method is faster for determining positive definite-ness for a 2-by-2 matrix, M.

matrix is $1 - 4 < 0$. Therefore this matrix is *neither positive nor negative definite.*

(b)

$$\begin{pmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

The first leading principal minor is 7, which is > 0. The second leading principal minor is the matrix itself. The determinant of this matrix is $7*1-\sqrt{3}*\sqrt{3} > 0$. Therefore this matrix is *positive definite.*

(c)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The first leading principal minor is 1, which is > 0. The second leading principal minor is the matrix itself. The determinant of this matrix is $1 * 1 - 1 * 1 = 0$. Therefore this matrix is *positive semi-definite* (but not positive definite).

(d)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of a diagonal matrix are the diagonal entries. These are all > 0. Therefore this matrix is *positive definite.*

3. (a) Show that the rate of convergence of the sequence $x^k = \frac{2k}{4^k+k^4+1} \to 0$ is linear.

Solution. We want to show that we can find a constant, c, such that $||x^{k+1} - x^*|| \le c||x^k - x^*||$.

$$||x^{k+1} - x^*|| = \frac{2(k+1)}{4^{k+1} + (k+1)^4 + 1},$$

and

$$||x^k - x^*|| = \frac{2k}{4^k + k^4 + 1}.$$

So the ratio is

$$\begin{aligned} \frac{||x^{k+1} - x^*||}{||x^k - x^*||} &= \frac{2(k+1)(4^k + k^4 + 1)}{2k(4^{k+1} + (k+1)^4 + 1)} \\ &= \frac{k4^k + k^5 + k + 4^k + k^4 + 1}{k4^{k+1} + k(k+1)^4 + k}. \end{aligned}$$

If we divide both the numerator and denominator by $k4^k$, then we can see that the limit of this expression as $k \to \infty$ is $\frac{1}{4}$. Therefore, for large enough k, the ratio of the normed errors is less than $\frac{1}{2}$, say. Taking $c = \frac{1}{2}$ shows that the sequence converges linearly (we can also use any c strictly greater than $\frac{1}{4}$).

(b) What is the rate of convergence of the sequence $x^k = \frac{2k^2-3k+8}{2k^2+7k-2} \to 1$?
Solution. We want to find the ratio of the error terms $x^{k+1} - 1$ and $x^k - 1$. First observe that:

$$\begin{aligned} ||x^k - 1|| &= \left| \frac{2k^2 - 3k + 8}{2k^2 + 7k - 2} - 1 \right| \\ &= \left| \frac{-10k + 10}{2k^2 + 7k - 2} \right|. \end{aligned}$$

Then

$$\begin{aligned} \frac{||x^{k+1} - 1||}{||x^k - 1||} &= \left| \frac{\frac{-10(k+1)+10}{2(k+1)^2+7(k+1)-2}}{\frac{-10k+10}{2k^2+7k-2}} \right| && (1) \\ &= \left| \frac{(2k^2 + 7k - 2)(-10k)}{(2k^2 + 11k + 7)(-10k + 10)} \right| && (2) \\ &\to 1. && (3) \end{aligned}$$

Therefore, we cannot find any number c strictly smaller than 1 for which $||x^{k+1} - 1|| \leq c||x^k - 1||$. Therefore the sequence converges slower than linearly. (It clearly does not converge superlinearly or quadratically, as they both imply linear convergence.)