

620-361 Operations Research Techniques and Algorithms

Practice Class 4

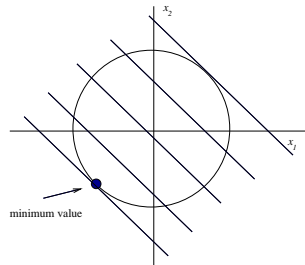
You are playing cards with your friends when you notice that the floor of the room is slanted. The slant is sufficient to provide additional viewing of your friends cards according to the function $x_1 + x_2$. Although you can see the *optimum cheating position* and the *optimum honesty position* at the round table with radius $\sqrt{2}$, you decide that this is an easy example to practice the Lagrangian method you have been learning about in your numerical analysis course. You opt to seek the optimum honesty position.

1. Formally write down the model of the problem with one constraint, c_1 .

Solution. $\min x_1 + x_2$ subject to $c_1 : x_1^2 + x_2^2 = 2$.

2. Sketch the feasible region and indicate the optimal value.

Solution. The feasible region is on the line c_1 as it is an equality constraint.



3. Express the Lagrangian of the problem and solve (the long way) to find x_1^* , x_2^* and η^* for the optimum honesty position.

Solution. The Lagrangian of the problem can be expressed:

$$\mathcal{L} = x_1 + x_2 + \eta(x_1^2 + x_2^2 - 2).$$

This gives us:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 + 2\eta x_1 = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 + 2\eta x_2 = 0 \quad (2)$$

$$c_1 : x_1^2 + x_2^2 = 2 \quad (3)$$

We see from equation 1 and equation 2 that at the optimal point $x_1 = x_2$. Substituting this into equation 3 we see that there are two optimum values:

(1, 1) and (-1, -1). By back-substituting we see that for each of these points we find that the optimal Lagrange multiplier value is $\eta = -\frac{1}{2}$ for (1, 1) and $\eta = \frac{1}{2}$ for (-1, -1).

4. You realise there are two optimums. Calculate the Hessian of the Lagrangian to demonstrate which optimal value is a maximum or minimum.

Solution. The Hessian matrix for this problem is:

$$\begin{pmatrix} 2\eta & 0 \\ 0 & 2\eta \end{pmatrix}$$

So for the point (1, 1) we have

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is negative definite. Hence (1, 1) is a maximum point. For the point (-1, -1) we have

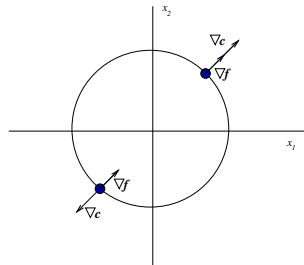
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is positive definite. Hence (-1, -1) is a minimum point.

5. Interpret the Lagrange multipliers¹.

Solution. For every Δ change in the radius of our table, we expect a $-\frac{\Delta^2}{2}$ change in the objective function for the honest player and a $\frac{\Delta^2}{2}$ change in the objective function for the cheating player. We could alternatively say that, for every Δ change in the right hand side of the constraint we expect a $\frac{\Delta}{2}$ change in the objective function (for the cheating optimum), but this is less interesting (although not incorrect). I chose to interpret the multipliers with respect to a physical aspect of the model - the radius of the table.

6. On your diagram, draw direction vectors for ∇f and ∇c_1 at both the minimum and the maximum.



¹Sometimes Lagrange multipliers have an interesting physical interpretation - we will investigate this more in the next week.