Consider the constrained nonlinear program

\[
\min_{x \in \mathbb{R}^2} f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2
\]

subject to \(x_1, x_2 \leq 0\).

1. You can see that this is a 2-D nonlinear optimisation problem with inequality constraints. You would like to solve it using the KKT conditions.

(a) Write down the Lagrangian and corresponding KKT conditions.

(b) Solve the system of equations.

2. Now suppose that you were unable to find the solution using the KKT conditions. You resort to a penalty method.

(a) Write down the \(l_2\)-penalty function \(P_k(x)\) with penalty parameter \(k\), and explain in general terms, how the \(\ell_2\)-penalty method approximates a solution to a constrained nonlinear program.

(b) Simplify \(P_k(x)\) when \(x_1 > 0\) and \(x_2 < 0\). Write down \(\nabla P_k(x)\) when \(x_1 > 0\) and \(x_2 < 0\).

(c) Find a stationary point \(x^k = (x^k_1, x^k_2)\) for \(P_k(x)\) such that \(x^k_1 > 0\) and \(x^k_2 < 0\). Write down the limit \(x^* = \lim_{k \to \infty} x^k\).

(d) Write down an estimate \(\lambda^k\) of the optimal multiplier vector, and find the limit \(\lambda^* = \lim_{k \to \infty} \lambda^k\).