620-361 Operations Research Techniques and Algorithms

Practice Class 6

Consider the constrained nonlinear program

$$\min_{x \in \Re^2} f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2$$
$$\text{subject to} \quad x_1, x_2 \leq 0.$$

1. You can see that this is a 2-D nonlinear optimisation problem with inequality constraints. You would like to solve it using the KKT conditions.

   (a) Write down the Lagrangian and corresponding KKT conditions.

   (b) Solve the system of equations.

   Solution

   (a) The Lagrangian is:

$$\mathcal{L}(x_1, x_2; \eta_1, \eta_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 + \eta_1 x_1 + \eta_2 x_2 \tag{1}$$

   (b) The KKT conditions are:

$$\begin{aligned} x_1 - 1 + \eta_1 &= 0 \\ x_2 + 1 + \eta_2 &= 0 \\ \eta_1 &\geq 0 \\ x_1 &\leq 0 \\ \eta_1 x_1 &= 0 \\ \eta_2 &\geq 0 \\ x_2 &\leq 0 \\ \eta_2 x_2 &= 0 \end{aligned}$$

   Clearly either $\eta_1 = 0 \Rightarrow x_1 = 1$ or $x_1 = 0 \Rightarrow \eta_1 = 1$. Since $x_1 \leq 0$, $x_1^* = 0$ and $\eta_1^* = 1$. Similarly, either $\eta_2 = 0 \Rightarrow x_2 = -1$ or $x_2 = 0 \Rightarrow \eta_2 = -1$. Since $\eta_2 \geq 0$, $\eta_2^* = 0$ and $x_2^* = -1$. So our KKT point is (0,-1) with corresponding KKT multipliers (1,0).

2. Now suppose that you were unable to find the solution using the KKT conditions. You resort to a penalty method.

   (a) Write down the l_2-penalty function $P_k(x)$ with penalty parameter k, and explain in general terms, how the ℓ_2-penalty method approximates a solution to a constrained nonlinear program.

   (b) Simplify $P_k(x)$ when $x_1 > 0$ and $x_2 < 0$. Write down $\nabla P_k(x)$ when $x_1 > 0$ and $x_2 < 0$.

   (c) Find a stationary point $x^k = (x_1^k, x_2^k)$ for $P_k(x)$ such that $x_1^k > 0$ and $x_2^k < 0$. Write down the limit $x^* = \lim_{k\to\infty} x^k$.

   (d) Write down an estimate λ^k of the optimal multiplier vector, and find the limit $\lambda^* = \lim_{k\to\infty} \lambda^k$.

Solution

(a) Penalty function:

$$P_k(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 + \frac{k}{2}[(x_1)_+]^2 + \frac{k}{2}[(x_2)_+]^2$$

The penalty method adds a "cost" proportional to the square of the amount by which each constraint $g_i(x) \leq 0$ is violated. As such, as the penalty parameter k increases, the cost of violating the constraints increases. The method then finds the **unconstrained** optimal value: $x^k = \arg\min P_k(x)$. Because $P_k(x) = f(x)$ when x is feasible, then x^k will (1) be closer to the feasible region as k increases, and (2) approach the constrained optimum x^* of $f(x)$.

(b) Let $x_1 > 0$ and $x_2 < 0$. Then

$$P_k(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 + \frac{k}{2}x_1^2$$
$$\nabla P_k(x) = \begin{pmatrix} (1+k)x_1 - 1 \\ x_2 + 1 \end{pmatrix}$$

(c) Setting $\nabla P_k(x) = 0$ for $x_1 > 0$ and $x_2 < 0$ gives $x_1^k = \frac{1}{1+k}, x_2^k = -1$, which is stationary for $P_k(x)$ in the required region. $x_1^* = \lim_{k\to\infty} x_1^k = 0$, so the limit point is $x^* = (0, -1)^T$.

(d) For the region $x_1 > 0$ and $x_2 < 0$,

$$\lambda^k = \begin{pmatrix} k\left(\frac{1}{1+k}\right)_+ \\ k(-1)_+ \end{pmatrix} = \begin{pmatrix} \frac{k}{1+k} \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$