620-361 Operations Research Techniques and Algorithms

Practice Class 7

1. Consider the nonlinear program (NLP):

$$\begin{array}{lll} \min_{x \in \Re^n} & f(x) & \\ \text{s.t.} & g_i(x) & \leq 0, \quad \forall i = 1, \ldots, m. \end{array}$$

   Prove that if (NLP) is a convex program with KKT point (x^*, λ^*), then x^* minimizes the Lagrangian function $L(x, \lambda^*)$, over all $x \in \Re^n$, i.e.,

$$L(x^*, \lambda^*) \leq L(x, \lambda^*)$$

   for all $x \in \Re^n$. [Hint: Observe that $L(x, \lambda^*)$ is a function of x, show that x^* is a stationary point of this function, and explain briefly why the function is convex.]

   Solution: By KKTa for (NLP), $\nabla_x L(x^*, \lambda^*) = 0$, so x^* is a stationary point of $L(x, \lambda^*)$. (NLP) convex implies $f(x)$ is convex and $g_i(x)$ is convex for each i. By KKTb, $\lambda_i^* \geq 0$, so $\lambda_i^* g_i(x)$ is also convex. Since the sum of convex functions is also convex it must be that $\sum_{i=1}^{m} \lambda_i^* g_i(x)$ is convex, and also $f(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x)$, since f is convex. But this is just $L(x, \lambda^*) = f(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x)$, by definition, so $L(x, \lambda^*)$ is a convex function of x. Now x^* a stationary point of a convex function means x^* is a global minimum, i.e.

$$L(x^*, \lambda^*) \leq L(x, \lambda^*)$$

   for all $x \in \Re^n$, as required.

2. Consider the program (LP):

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

   where $c \in \Re^n$, $A \in \Re^{n \times m}$ and $b \in \Re^m$.

   (a) Explain briefly why (LP) is a convex program. Write down the Lagrangian function for (LP), using the equality constraint in the form $h(x) = Ax - b = 0$.

Solution: The objective function and all constraints are affine, and so (LP) is a convex program. The Lagrangian function is

$$L(x, \lambda, \eta) = c^T x - \lambda^T x - \eta^T (Ax - b).$$

(b) Explain why the Lagrangian dual objective function for (LP) can be given by

$$\psi(\lambda, \eta) = \begin{cases} \eta^T b & \text{if } c^T - \lambda^T - \eta^T A = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Solution: The dual objective is

$$\begin{aligned} \psi(\lambda, \eta) &= \min_{x \in \Re^n} L(x, \lambda, \eta) \\ &= \min_{x \in \Re^n} \big(c^T x - \lambda^T x - \eta^T (Ax - b)\big) \\ &= \min_{x \in \Re^n} \big((c^T - \lambda^T - \eta^T A)x + \eta^T b\big) \\ &= \eta^T b + \min_{x \in \Re^n} (c^T - \lambda^T - \eta^T A)x. \end{aligned}$$

Now if any component of $c^T - \lambda^T - \eta^T A$ is negative, we can increase the corresponding component of x indefinitely, resulting in a value of $-\infty$. Similarly, if any component of $c^T - \lambda^T - \eta^T A$ is positive, we can decrease the corresponding component of x indefinitely, again resulting in a value of $-\infty$. Thus either $c^T - \lambda^T - \eta^T A = 0$ or the value of the dual objective function is $-\infty$.

(c) Write down the Lagrangian dual problem for (LP), and simplify if possible.

Solution:

$$\max_{\lambda \geq 0, \eta} \psi(\lambda, \eta) = \begin{cases} \max\limits_{\lambda \geq 0, \eta} & \eta^T b \\ \text{s.t.} & c^T - \lambda^T - \eta^T A = 0. \end{cases}$$

(d) Given $(\hat{\lambda}, \hat{\eta})$ with $\hat{\lambda} \geq 0$ and $c^T - \hat{\lambda}^T - \hat{\eta}^T A = 0$, what can you deduce about the optimal value of (LP)?

Solution: Since (LP) is convex, by weak duality it must be that the optimal value of (LP) is at least $\hat{\eta}^T b$.