1. Consider the nonlinear program (NLP):

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad \forall i = 1, \ldots, m.
\end{align*}
\]

Prove that if (NLP) is a convex program with KKT point \( (x^*, \lambda^*) \), then \( x^* \) minimizes the Lagrangian function \( L(x, \lambda^*) \), over all \( x \in \mathbb{R}^n \), i.e.,

\[
L(x^*, \lambda^*) \leq L(x, \lambda^*)
\]

for all \( x \in \mathbb{R}^n \). [Hint: Observe that \( L(x, \lambda^*) \) is a function of \( x \), show that \( x^* \) is a stationary point of this function, and explain briefly why the function is convex.]

**Solution:** By KKTa for (NLP), \( \nabla_x L(x^*, \lambda^*) = 0 \), so \( x^* \) is a stationary point of this function, and explain briefly why the function is convex.

By KKTb, \( \lambda_i^* \geq 0 \), so \( \lambda_i^* g_i(x) \) is also convex. Since the sum of convex functions is also convex it must be that \( \sum_{i=1}^{m} \lambda_i^* g_i(x) \) is convex, and also \( f(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x) \), since \( f \) is convex. But this is just \( L(x, \lambda^*) = f(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x) \), by definition, so \( L(x, \lambda^*) \) is a convex function of \( x \). Now \( x^* \) a stationary point of a convex function means \( x^* \) is a global minimum, i.e.

\[
L(x^*, \lambda^*) \leq L(x, \lambda^*)
\]

for all \( x \in \mathbb{R}^n \), as required.

2. Consider the program (LP):

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

where \( c \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times m} \) and \( b \in \mathbb{R}^m \).

(a) Explain briefly why (LP) is a convex program. Write down the Lagrangian function for (LP), using the equality constraint in the form \( h(x) = Ax - b = 0 \).
Solution: The objective function and all constraints are affine, and so (LP) is a convex program. The Lagrangian function is

\[ L(x, \lambda, \eta) = c^T x - \lambda^T x - \eta^T (Ax - b) \]

(b) Explain why the Lagrangian dual objective function for (LP) can be given by

\[ \psi(\lambda, \eta) = \begin{cases} 
\eta^T b & \text{if } c^T - \lambda^T - \eta^T A = 0 \\
-\infty & \text{otherwise.}
\end{cases} \]

Solution: The dual objective is

\[ \psi(\lambda, \eta) = \min_{x \in \mathbb{R}^n} L(x, \lambda, \eta) = \min_{x \in \mathbb{R}^n} (c^T x - \lambda^T x - \eta^T (Ax - b)) = \min_{x \in \mathbb{R}^n} ((c^T - \lambda^T - \eta^T A)x + \eta^T b) = \eta^T b + \min_{x \in \mathbb{R}^n} (c^T - \lambda^T - \eta^T A)x. \]

Now if any component of \( c^T - \lambda^T - \eta^T A \) is negative, we can increase the corresponding component of \( x \) indefinitely, resulting in a value of \(-\infty\). Similarly, if any component of \( c^T - \lambda^T - \eta^T A \) is positive, we can decrease the corresponding component of \( x \) indefinitely, again resulting in a value of \(-\infty\). Thus either \( c^T - \lambda^T - \eta^T A = 0 \) or the value of the dual objective function is \(-\infty\).

(c) Write down the Lagrangian dual problem for (LP), and simplify if possible.

Solution:

\[ \max_{\lambda \geq 0, \eta} \psi(\lambda, \eta) = \begin{cases} 
\max_{\lambda \geq 0, \eta} \eta^T b & \text{if } c^T - \lambda^T - \eta^T A = 0 \\
s.t. & \end{cases} \]

(d) Given \((\hat{\lambda}, \hat{\eta})\) with \(\hat{\lambda} \geq 0\) and \(c^T - \hat{\lambda}^T - \hat{\eta}^T A = 0\), what can you deduce about the optimal value of (LP)?

Solution: Since (LP) is convex, by weak duality it must be that the optimal value of (LP) is at least \(\hat{\eta}^T b\).